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Theory of Errors and Least Squares Adjustment

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To the memory of my father

Fuguan Fan (1928-1996)

who loved and inspired a geodesist

Preface

Theory of errors has been a well defined subject within geodesy, surveying and photogrammetry. Presently there are several masterpieces on this subject, such as the famous works by Bjerhammar (1973), Mikhail (1976) and Koch (1980), just to mention a few of them. However, the author still feels the need for a middle-level textbook in this field. A textbook that balances between practical applications and pure mathematical treatments; A textbook that integrates classical adjustment methods with modern developments in geodesy, surveying and photogrammetry.

To meet the above need, an attempt was started in 1995 that has resulted in this compendium. Naturally, it is neither the author's ambition nor within his capability to challenge those great works mentioned above. The compendium will primarily be used for courses given at the Royal Institute of Technology (KTH).

The compendium consists of seven chapters. Chapter 1 deals with basic concepts in theory of errors, such as standard error and error propagation, error ellipse and error ellipsoid, linear equation systems as well as some elementary concepts of statistical analysis. Chapter 2 is devoted to the classical condition adjustment method, including condition adjustment in groups and condition adjustment with unknowns. The method of adjustment by elements is treated in Chapter 3, where adjustment by elements with constraints and sequential adjustment have also been described. Chapters 4-7 deal with diverse topics based on recent developments in theory of errors. These topics include generalized matrix inverses and their applications in least squares adjustment; a posteriori estimation of variance-covariance components; detection of gross and systematic errors; and finally prediction and filtering in linear dynamic systems.

The essential prerequisite for the compendium is a familiarity with linear algebra, mathematical statistics and basic concepts in surveying. In other words, it is assumed that students have already acquired background knowledge in mathematics and surveying. Therefore, efforts have been made to limit discussions on pure mathematical or surveying subjects.

In order to keep mathematical derivations brief and elegant, matrix notations have been used exclusively throughout the compendium. Several old concepts in theory of errors (e.g. the classical Gauss-Doolittle's table for solving normal equations) have, in the author's opinion, become out of date and thus been excluded.

To help readers better understand the theoretical concepts, a number of numerical examples (mostly originating from geodesy and surveying) are provided. For the sake of simplicity, most numerical examples are constructed so that only ideally simple numbers are involved and that they can be solved without needing to use calculators and computers. For those examples with more realistic data, the presented numerical results are obtained based on the author's own programming on an Intel Pentium PC using Lahey Fortran 77 Compiler (version 5.1). Minor decimal differences might occur if the same examples are calculated using other hardware and software.

The author wishes to acknowledge all the help and encouragement from his colleagues and students. Special thanks go to TeknL *George Stoimenov* for proof-reading the manuscript and Mr *Hossein Nahavandchi* for helping draw some of the figures. Of particular benefit have been my students in the classes MK-92, TL-93, TL-94 and TL-95, who have worked through the raw materials as the compendium evolved. To all of them, I express my sincere thanks !

Stockholm, August 1997. *Huaan Fan*

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Chapter 1

Fundamentals of Theory of Errors

Science and engineering often involves measurements of different types . In geodesy and surveying, geometrical quantities (such as angles, distances, heights, etc.) or physical quantities (e.g. gravity) are directly measured, producing large amounts of data which need to be processed. To some extent, a surveying project may be considered as a data production process, from data collection, data processing, to final presentation (graphically and/or digitally).

Due to human limitations, imperfect instruments, unfavourable physical environment and improper measurement routines, which together define the *measurement condition*, all measurement results most likely contain errors. One can discover the existence of measurement errors in different ways. If we repeat the same measurement several times, we will normally get different results due to measurement errors. Another way to discover errors is to check whether the obtained measurement results satisfy some geometrical or physical relations which may exist. For example, one may check whether the sum of three measured angles of a plane triangle is equal to the theoretical value, 180 degrees.

Normally, one may distinguish three types of errors: *systematic errors, gross errors* and *random errors.*

Systematic errors are errors which follow certain physical or mathematical rules and often affect surveying results systematically. The causes of this kind of errors can be the instruments used, physical environment in which measurements are made, human factors and measurement routines. To avoid or reduce systematic errors, one may *(a)* calibrate carefully instruments before field work starts; *(b)* design and use suitable measurement routines and procedures which can reduce or eliminate possible systematic errors; *(c)* if possible, correct measurement results afterwards. One example of systematic errors is the constant error of $-5\ cm$ for a tape. This constant error will cause a systematic error to all distance measurements by this tape. Another example is the tropospheric effect on GPS satellite signal transmission. To reduce the tropospheric effect on GPS measurements, one may measure both $L1$ and $L2$ frequencies of GPS signals so that the tropospheric effects can be reduced through dual frequency combinations.

Gross errors are errors due to human mistakes, malfunctioning instruments or wrong measurement methods. Gross errors do not follow certain rules and normally cannot be treated by statistical methods. In principle, gross errors are not permitted and should be avoided by surveyors$'$ carefulness and control routines. For example, it can happen that a surveyor might write $50^0\ 32'\ 50.9''$ in his field observation protocol when the actual reading on the theodolite is $50^0\ 32'\ 5.9''$. If the surveyor is highly concentrated during the measurement, he or she may be able to avoid this kind of blunders. On the other hand, if he or she measures the direction by both right circle and left circle, or measure the same direction by more than one complete rounds, the mistake can easily be discovered. Gross errors are also called *blunders* or *outliers.*

Random errors or *stochastic errors* are errors which behave randomly and affect the measurements in a non-systematic way. Random errors can be caused by human factors, instrument errors, physical environment and measurement routines. They can be reduced if the total measurement condition has been improved. The primary study object of theory of errors is just random errors. Probability theory and mathematical statistics is the science which specializes in studies of random (or stochastic) events, variables and functions. It will serve as the theoretical base for our treatment of the random measurement errors. In Chapter 6, we will briefly discuss how to detect gross errors and systematic errors.

Based on analysis of large amounts of available observation data (e.g. thousands of triangular misclosures in geodetic triangulation networks), it has been found that random errors, though non-systematic, show certain statistical characteristics. If a set of errors $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ have occurred under (roughly) the same measurement condition, then the following statistical characteristics have been discovered:

- The arithmetic mean of ε_i approaches zero when the number n of observations approaches infinity:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \varepsilon_i}{n} = 0 \tag{1.1}$$

- Positive errors and negative errors with same magnitude occur roughly at equal frequency;
- Errors of smaller magnitude occur more often than errors of larger magnitude;
- Under specific measurement condition, the absolute magnitude of errors is within some limit.

To reduce measurement errors and their effects on the final surveying results, one need to improve the overall measurement condition. As errors are impossible to avoid completely, it is natural to do redundant measurements, both to discover the existence of errors and to increase accuracy and reliability of the final results. When measurement errors are present and redundant measurements are made, there will exist inconsistency or ″contradiction″ among measurements, also called *misclosure*. One of the tasks of geodetic and photogrammetric computations is to get rid of misclosures among measurements in an optimal way according to some estimation criteria (such as the least squares principle). These criteria should naturally be based on the property of the measurement errors. Traditionally, the work or process to eliminate misclosures among measurements and obtain the best results out of available measurement data is called *adjustment*.

Another important task of theory of errors is to assess the quality of observations and results derived from observations. The quality of observations concern three related but different aspects: *precision*, *accuracy* and *reliability*. *Precision* describes the degree of repeatability of the observations and is an internal measure of random errors and their effects on the quality of observation. *Accuracy* is a measure of agreement between the observed value and the correct value. It is influenced not only by random errors but also, more significantly, by systematic or other non-random errors. *Reliability* concerns the capability of observations against gross and systematic errors. Observations or surveying methods with high reliability can easily detect gross and systematic errors, and are said to be *robust*. In general, redundant observations can improve the precision, accuracy and reliability of the observations as well as the derived results.

Theory of errors and least squares adjustment is an important subject within the geomatics programme offered at KTH. This is due to the fact that surveying and mapping (or production of spatial data) often requires mathematical processing of measurement data. Furthermore, the general methodology of spatial data processing is essentially the same as that for data processing in other science and engineering fields, even though data collection procedures and data types can be different. Theory of errors is related to and comparable with what is called *estimation theory* used in automatic control and signal processing. Therefore, studying theory of errors can be helpful for solving general data processing problems in other scientific and engineering fields.

The theoretical foundation of theory of errors is *probability theory and mathematical statistics*. As numerical computations are frequently involved, *numerical methods* in general and *matrix algebra* in particular are useful tools. Other useful mathematical tools include *linear programming*, *optimization theory* and *functional analysis* (theory of vector spaces).

1.1 Standard Errors and Weights

Standard errors and weights are probably the most elementary and fundamental concepts in theory of errors. They are used as ″*index numbers*″ to describe the accuracy of the measured and derived quantities.

Let $\widetilde{\ell}$ and ℓ denote the true value and the measured value of a quantity, respectively. The *true error* ε of measurement ℓ is defined as the measured value minus the true value :

$$\varepsilon = \ell - \widetilde{\ell} \tag{1.2}$$

ε is also called the *absolute (true) error* of ℓ, while the *relative error* for ℓ can be defined by:

$$\gamma = \left|\frac{\varepsilon}{\ell}\right| = \left|\frac{\ell - \widetilde{\ell}}{\ell}\right| \tag{1.3}$$

γ is a more suitable index for describing the error of ℓ in the case when the error ε tends to increase with the absolute magnitude of ℓ. A typical example is the error of distance measurements in geodesy and surveying. Though two distances, at 1000 and 10 meters respectively, might have equal true errors of 1 cm, the quality of the first measurement would be much better than that of the second one, as their relative errors are different: 1 : 100000 for the first distance and 1 : 1000 for the second one.

Both absolute error ε and relative error γ defined above are describing an individual measurement error. Most measurement errors are random errors that behave in a random way. Therefore, in practice it is very difficult or impossible to describe each individual error which occurs under a specific measurement condition (i.e. under a specific physical environment, at a specific time, with a specific surveyor, using a specific instrument, etc). However, for simplicity we often prefer to use one or several simple index numbers to judge the quality of the obtained measurements (how good or how bad the measurements are). An old and well known approach is to define and use statistical error indices.

1.1.1 Standard Errors

The statistical way of thinking does not try to describe each individual error separately, rather try to judge the collective property of the same group of errors. It assumes that it is the total measurement condition (i.e. measurement instruments used, physical environment of field survey, professional skills of surveyors, etc.) that determines the quality of the obtained observations. In other words, different observations with true errors different in size and sign will still be regarded as having the same quality or accuracy, *if* they are made under same or similar measurement condition.

Standard error is an error index defined in mathematical statistics and used in theory of errors to describe the *average* observation errors coming from the same (or similar) measurement condition. Standard error is also called *standard deviation*, or *mean-square-root error*. Below, we will define standard errors in an intuitive way, while a more rigorous definition based on probability theory will be given in Subsection 1.1.3.

Assume that a quantity with true value $\widetilde{\ell}$ has been measured n times independently under the same measurement condition, i.e. with same accuracy. Denoting the $i - th$ observed value by ℓ_i $(i = 1, 2, 3, \cdots n)$ and its (true) error by ε_i , i.e. $\varepsilon_i = \ell_i - \widetilde{\ell}$, the *theoretical standard error* (σ) of this group of observations is then defined by:

$$\sigma^2 = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \varepsilon_i^2}{n} \tag{1.4}$$

When the number of measurements is limited to a finite number n, one gets only an estimate of σ:

$$\widehat{\sigma}^2 = \frac{\sum_{i=1}^{n} \varepsilon_i^2}{n} \tag{1.5}$$

In probability theory, σ^2 is called the *variance* of the random variable ε (Cf Subsection 1.1.3).

Example 1.1

Let ℓ_1, ℓ_2 and ℓ_3 denote the observed values of three internal angles in a plane triangle. The true error of the sum of these three internal angles can be calculated from the observed angles:

$$w = \ell_1 + \ell_2 + \ell_3 - 180^0 \tag{1.6}$$

w defined above is called the *misclosure of the triangle.* Assume that in a geodetic triangulation network, 30 triangles have been observed with the triangular misclosures (w_i , $i = 1, 2, \cdots, 30$) as listed in Table 1.1 :

Table 1.1: List of Triangular Misclosures

i	w_i ($''$)	i	w_i ($''$)	i	w_i ($''$)
1	+1.5	11	-2.0	21	-1.1
2	+1.0	12	-0.7	22	-0.4
3	+0.8	13	-0.8	23	-1.0
4	-1.1	14	-1.2	24	-0.5
5	+0.6	15	+0.8	25	+0.2
6	+1.1	16	-0.3	26	+0.3
7	+0.2	17	+0.6	27	+1.8
8	-0.3	18	+0.8	28	+0.6
9	-0.5	19	-0.3	29	-1.1
10	+0.6	20	-0.9	30	-1.3

If all triangles are observed independently of each other, the standard error of the misclosures can be estimated using (1.5):

$$\widehat{\sigma}_w{}^2 = \frac{1}{30}\sum_{i=1}^{30} w_i^2 = \frac{25.86}{30} \qquad (''^2) \tag{1.7}$$

or:

$$\widehat{\sigma}_w = 0.93''$$

If all three angles in each triangle are uncorrelated and of equal accuracy (i.e. having equal standard error), the standard error of each angle observation can be estimated as:

$$\widehat{\sigma} = \frac{\widehat{\sigma}_w}{\sqrt{3}} = 0.54''$$

The last formula can be obtained by applying error propagation law on Eq.(1.6) (Cf Section 1.2).

In practice, the observed angles of a triangle and angles of different triangles are most likely correlated with each other. Therefore, Eq. (1.7) is not a rigorous formula for estimating the standard error of angle measurements. Nevertheless, it still provides us with a measure on the quality of angle measurements in geodetic triangulation.

1.1.2 Weights and Unit-Weight Standard Error

Sometimes, what we are most interested in is not the absolute magnitude of observation errors, rather the relative accuracy of observations. According to the relative accuracy of different observations, we assign each observation a positive number, called *weight.* The smaller an observation error is, the more accurate the observation will be and consequently the bigger weight the observation should have. Therefore, in theory of errors the weight p_i of an observation ℓ_i is defined to be *inversely proportional to the variance* σ_i^2 (standard error squared) of ℓ_i:

$$p_i = \frac{c_0}{\sigma_i^2} \tag{1.8}$$

where c_0 is an arbitrary positive number.

Based on practical experiences, the following empirical weights are used in geodesy and surveying:

- *levelling:* $p_i = c_0$ divided by the length of the levelling line
- *distance measurement:* $p_i = c_0$ divided by the distance or distance squared
- *direction (angle) measurement*: $p_i =$ number of whole rounds measured divided by c_0

It should be noted that outside the field of theory of errors, the concept of weights may be based on factors other than standard errors. For example, when calculating an "average" price index of a stock market, the total market value of each share (or its daily turn-out) may be used as the basis to define the weight.

Assume that there is an observation with standard error σ_0 and weight $p_0 = 1$. Eq.(1.8) then gives: $c_0 = {\sigma_0}^2$, *i.e.* the arbitrary constant c_0 is equal to the variance of the observation with unit weight $p = 1$. Therefore, σ_0 is called the standard error of unit weight or *unit-weight standard error* (in Swedish: *grundmedelfel*) and $c_0 = {\sigma_0}^2$ is called *variance factor*. Then (1.8) can be written as:

$$p_i = \frac{\sigma_0^2}{\sigma_i^2} \tag{1.9}$$

It is not difficult to find out the geodetic meaning of c_0 in the above empirical weights:

- *levelling:* $c_0 =$ length of the levelling line with weight 1
- *distance measurement:* $c_0 =$ distance or squared distance of weight 1
- *direction (angle) measurement:* $c_0 =$ number of whole rounds by which the direction (angle) of unit weight is measured

Naturally, observations of different weights will have different standard errors. For observation ℓ_i with true error ε_i and weight p_i $(i = 1, 2, 3, \cdots, n)$, one can calculate the *theoretical* unit-weight standard error as:

$$\sigma_0^2 = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} p_i \varepsilon_i \varepsilon_i \tag{1.10}$$

For a finite n , we obtain instead an *estimated* unit-weight standard error:

$$\widehat{\sigma}_0^2 = \frac{1}{n} \sum_{i=1}^{n} p_i \varepsilon_i \varepsilon_i \tag{1.11}$$

From (1.9), we can get the theoretical variance σ_i^2 and the estimated variance $\widehat{\sigma}_i^2$ for observation ℓ_i of weight p_i:

$$\sigma_i^2 = \frac{\sigma_0^2}{p_i}\ , \quad \widehat{\sigma}_i^2 = \frac{\widehat{\sigma}_0^2}{p_i} \tag{1.12}$$

In practice the true value $\widetilde{\ell}_i$ $(i = 1, 2, 3, \cdots, n)$ of observation ℓ_i is often unknown and thus formulas (1.4) and (1.5) or (1.10) and (1.11) cannot be applied directly. A practical solution is to find first an optimal estimate $\widehat{\ell}_i$ for ℓ_i based on some theoretical criteria (e.g. the least squares principle) and then obtain an estimate for the unit-weight standard error as:

$$\widehat{\sigma}_0^2 = \frac{1}{f} \sum_{i=1}^{n} p_i \widehat{\varepsilon}_i \widehat{\varepsilon}_i \tag{1.13}$$

where :

$p_i =$ weight of ℓ_i ;

$\widehat{\varepsilon}_i = \ell_i - \widehat{\ell}_i$ (= estimated error of ℓ_i);

f = number of redundant observations (statistical degrees of freedom)

The estimated unit-weight standard error given by (1.13) is solely dependent on the estimate $\widehat{\ell}_i$. Only good estimate $\widehat{\ell}_i$ can lead to meaningful estimate $\widehat{\sigma}_0$.

1.1.3 Variance-Covariance Matrix and Cofactor Matrix

For a random variable ε, its probability *distribution function* $F(x)$ and probability *frequency function* $f(\varepsilon)$ are associated by equation:

$$P(\varepsilon \leq x) = F(x) = \int_{-\infty}^{x} f(\varepsilon) d\varepsilon \qquad (-\infty < x < +\infty) \tag{1.14}$$

where $P(\varepsilon \leq x)$ denotes the probability that $\varepsilon \leq x$. $f(\varepsilon)$ is also known as the *density function* of ε.

Eq. (1.14) indicates that geometrically $F(x)$ is equal to the area between the horizontal ε-axis and curve $f(\varepsilon)$ within the interval $(-\infty,\, x)$. The frequency function characterizes the changing rate of the probability. This can be seen more clearly from the following relation which is obtained by differentiating both sides of Eq.(1.14) with respect to x :

$$\frac{\partial F(x)}{\partial x} = f(x) \tag{1.15}$$

Mathematically, $f(x)$ or $F(x)$ is able to describe the complete analytical properties of random variable ε. Practically one can use several simple index numbers, called *characteristic values*, to describe the most important (but not necessarily complete) properties of the random variable. *Mathematical expectation* and *variance* are two such characteristic numbers. The mathematical expectation of random variable ε , denoted as $E(\varepsilon)$, is defined as the average value of ε weighted against the probability over the whole definition interval:

$$E(\varepsilon) = \int_{-\infty}^{+\infty} \varepsilon\ f(\varepsilon) d\varepsilon \tag{1.16}$$

The variance of ε, denoted as $var(\varepsilon)$ or σ^2, is defined as:

$$var(\varepsilon) = \sigma^2 = E\left\{[\varepsilon - E(\varepsilon)]^2\right\} = \int_{-\infty}^{+\infty} [\varepsilon - E(\varepsilon)]^2 \cdot f(\varepsilon) \cdot d\varepsilon \tag{1.17}$$

If ε denotes a geodetic measurement error, $E(\varepsilon)$ may be regarded as the theoretically true value of ε. When there is no systematical error or gross error in our measurements, it is naturally to believe that the expectation of measurement errors should be equal to zero, *i.e.* $E(\varepsilon) = 0$. On the other hand, σ^2 represents the "average" deviation of ε from the theoretical mean value $E(\varepsilon)$. This explains why σ is also called the *standard deviation* of ε.

Using the mathematical expectation operator $E(\cdot)$, the *covariance* between two random variables ε_1 and ε_2 can be defined:

$$\sigma_{12} = E\left\{[\varepsilon_1 - E(\varepsilon_1)][\varepsilon_2 - E(\varepsilon_2)]\right\} \tag{1.18}$$

If the variances of ε_1 and ε_2 are σ_1^2, σ_2^2, respectively, and the covariance between them is σ_{12}, the correlation coefficient between them is defined as :

$$\rho_{12} = E\left[\frac{\varepsilon_1 - E(\varepsilon_1)}{\sigma_1}\frac{\varepsilon_2 - E(\varepsilon_2)}{\sigma_2}\right] = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2} \tag{1.19}$$

It can be easily shown :

$$-1 \leq \rho_{12} \leq +1 \tag{1.20}$$

Now let us look at a random vector $\underset{n\cdot 1}{\varepsilon}$ of n random variables ε_i $(i = 1,\ 2,\ \cdots,\ n)$:

$$\underset{n\cdot 1}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix} \tag{1.21}$$

Assume that ε_i has zero expectation and that its variance and covariance with ε_j are as follows:

$$\left.\begin{array}{l} E(\varepsilon_i) = 0 \\ E\left\{[\varepsilon_i - E(\varepsilon_i)]^2\right\} = E(\varepsilon_i^2) = \sigma_i^2 \qquad (i,\ j = 1, 2, 3, \cdots, n) \\ E\left\{[\varepsilon_i - E(\varepsilon_i)]\left[\varepsilon_j - E(\varepsilon_j)\right]\right\} = E\left(\varepsilon_i \cdot \varepsilon_j\right) = \sigma_{ij} \end{array}\right\} \tag{1.22}$$

The *variance-covariance matrix* $C_{\varepsilon\varepsilon}$ of random vector ε is then defined as:

$$\underset{n\cdot n}{C_{\varepsilon\varepsilon}} = E\left\{[\varepsilon - E(\varepsilon)]\,[\varepsilon - E(\varepsilon)]^\top\right\} = E\left\{\varepsilon\varepsilon^\top\right\} = E\left(\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_n \end{bmatrix} [\varepsilon_1,\ \varepsilon_2,\ \cdots,\ \varepsilon_n]\right)$$

$$= \begin{bmatrix} E\left(\varepsilon_1^2\right) & E\left(\varepsilon_1\varepsilon_2\right) & \cdots & E\left(\varepsilon_1\varepsilon_n\right) \\ E\left(\varepsilon_2\varepsilon_1\right) & E\left(\varepsilon_2^2\right) & \cdots & E\left(\varepsilon_2\varepsilon_n\right) \\ \cdots & \cdots & \cdots & \cdots \\ E\left(\varepsilon_n\varepsilon_1\right) & E\left(\varepsilon_n\varepsilon_2\right) & \cdots & E\left(\varepsilon_n^2\right) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix} \tag{1.23}$$

If there is a matrix $\underset{n\cdot n}{Q_{\varepsilon\varepsilon}}$ which satisfies:

$$\underset{n\cdot n}{C_{\varepsilon\varepsilon}} = \sigma_0^2 \underset{n\cdot n}{Q_{\varepsilon\varepsilon}} \tag{1.24}$$

$\underset{n\cdot n}{Q_{\varepsilon\varepsilon}}$ is called the *cofactor matrix* of random vector ε:

$$\underset{n\cdot n}{Q_{\varepsilon\varepsilon}} = \frac{1}{\sigma_0^2} \cdot \underset{n\cdot n}{C_{\varepsilon\varepsilon}} \tag{1.25}$$

The inverse matrix of $\underset{n\cdot n}{Q_{\varepsilon\varepsilon}}$, denoted as $\underset{n\cdot n}{P_{\varepsilon\varepsilon}}$, is called the *weight matrix* of ε :

$$P_{\varepsilon\varepsilon} = \left(Q_{\varepsilon\varepsilon}\right)^{-1} \tag{1.26}$$

When vector ε consists of only *uncorrelated* components $\varepsilon_1,\ \varepsilon_2,\ \cdots,\ \varepsilon_n$, the weight matrix $P_{\varepsilon\varepsilon}$ will be a diagonal matrix and the $i - th$ diagonal element of $P_{\varepsilon\varepsilon}$ $(i = 1,\ 2,\ 3, \cdots,\ n)$ denotes the weight of the corresponding element ε_i. However, if $\varepsilon_1,\ \varepsilon_2,\ \cdots,\ \varepsilon_n$ are correlated, $P_{\varepsilon\varepsilon}$will be a non-diagonal matrix whose diagonal elements are not the weights of the corresrponding elements. The weight of each ε_i can be calculated from the definition formulas (1.9). See **Example 1.5** at the end of the next section.