Global gravitational models (GGM)

A series expansion of the earth's gravitational potential by spherical harmonic functions (with respect to a sphere)

$$V(r,\overline{\phi},\lambda) = \frac{GM}{r} + \frac{GM}{a}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}(\overline{A}_{nm}\cos m\lambda + \overline{B}_{nm}\sin m\lambda)\overline{P}_{nm}(\sin\overline{\phi})$$

V:	gravitational potential of the earth
GM:	earth's gravitational constant (M = total mass)
a:	equatorial radius of the earth ellipsoid
A_{nm}, B_{nm}:	geopotential coefficients of the earth
P_{nm}:	*Legendre* functions of degree n and order m

Global gravitational models (GGM)

- Spherical harmonic series of earth's gravitational potential V
  - ✓ from *mathematics*, Laplace's equation
  - ✓ from *physics*, Newton's gravitational law
  - ✓ what are the meaning of zero-, 1st, 2nd degree terms ?
- Spherical harmonic series of other gravity field quantities
- Estimation of geopotential coefficients
- Available GGMs and applications for geoid determination

Laplace's equation

- in rectangular coordinates (x,y,z)

$$\Delta(V) \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- in spherical coordinates $(r, \bar{\phi}, \lambda)$:

V is called a **harmonic function**

$$r^2 \frac{\partial^2 V}{\partial r^2} + 2r \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \bar{\phi}^2} - \tan \bar{\phi} \frac{\partial V}{\partial \bar{\phi}} + \frac{1}{\cos^2 \bar{\phi}} \frac{\partial^2 V}{\partial \lambda^2} = 0$$

Example of a harmonic function

$$\ell = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\frac{\partial \ell}{\partial x} = \frac{2(x - x')}{2\ \ell} = \frac{x - x'}{\ell}$$

$$f(x, y, z) = \frac{1}{\ell}$$

$$\left(\frac{u}{v}\right)' = \frac{u'\ v - u\ v'}{v^2}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\ell}\right) = -\frac{1}{\ell^2}\frac{\partial \ell}{\partial x} = -\frac{x - x'}{\ell^3}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\ell}\right) = \frac{\partial}{\partial x}\left(-\frac{x - x'}{\ell^3}\right) = -\frac{1 \cdot \ell^3 - (x - x')\ 3\ell^2\ \frac{x - x'}{\ell}}{\ell^6} = \frac{\ell^2 - 3 \cdot (x - x')^2}{\ell^5}$$

Example of a harmonic function

$$\frac{\partial}{\partial y^2}(\frac{1}{\ell}) = \frac{\ell^2 - 3\cdot(y-y')^2}{\ell^5}$$

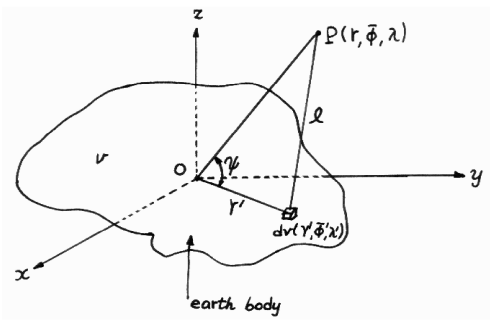
$$\frac{\partial}{\partial z^2}(\frac{1}{\ell}) = \frac{\ell^2 - 3\cdot(z-z')^2}{\ell^5}$$

$$\Delta(\frac{1}{\ell}) = \frac{\partial^2}{\partial x^2}(\frac{1}{\ell}) + \frac{\partial^2}{\partial y^2}(\frac{1}{\ell}) + \frac{\partial^2}{\partial z^2}(\frac{1}{\ell})$$

$$= \frac{3\cdot\ell^2 - 3\cdot\{(x-x')^2+(y-y')^2+(z-z')^2\}}{\ell^5} = \frac{3\ell^2 - 3\ell^2}{\ell^5} = 0$$

→ *f(x,y,z), reciprocal distance*, is a harmonic function

Laplace's equation



$$\ell = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\Delta(\frac{1}{\ell}) \;=\; 0$$

$$\Delta(V) = \Delta\left(G\iiint_v \frac{\rho dv}{\ell}\right) =$$

$$= G\iiint_v \Delta(\frac{1}{\ell})\,\rho dv \;=\; 0$$

→ The external gravitational potential V is always a harmonic function

General solution of Laplace's equation

$$\Delta(V) = 0 \qquad r^2\frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \bar{\phi}^2} - \tan\bar{\phi}\frac{\partial V}{\partial \bar{\phi}} + \frac{1}{\cos^2\bar{\phi}}\frac{\partial^2 V}{\partial \lambda^2} = 0$$

$(r, \bar{\phi}, \lambda)$

$$V(r,\bar{\phi},\lambda) = f_1(r)\cdot f_2(\bar{\phi})\cdot f_3(\lambda)$$

$$\lim_{r\to\infty} V(r,\bar{\phi},\lambda) = 0$$

$$f_1(r) = \frac{1}{r^{n+1}} \text{ or } r^n \qquad (n = 0, 1, 2, \cdots)$$

$$f_2(\bar{\phi}) = P_{nm}(\sin\bar{\phi}) \qquad (n = 0, 1, 2, \cdots,\ m = 0, 1, 2, \cdots, n-1, n)$$

$$f_3(\lambda) = \cos m\lambda \text{ or } \sin m\lambda \qquad (m = 0, 1, 2, \cdots, n-1, n)$$

General solution of Laplace's equation

$$V(r,\bar{\phi},\lambda) = \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=0}^{n}(a_{nm}\cos m\lambda + b_{nm}\sin m\lambda)\ P_{nm}(\sin\bar{\phi})$$

- introducing new notations:

$$Y_{nm}(\bar{\phi},\lambda) = \begin{cases} \cos m\lambda P_{n|m|}(\sin\bar{\phi}) & \text{for } m \le 0 \\ \sin m\lambda P_{nm}(\sin\bar{\phi}) & \text{for } m > 0 \end{cases}$$

$$A_{nm} = \begin{cases} a_{n|m|} & \text{for } m \le 0 \\ b_{nm} & \text{for } m > 0 \end{cases}$$

$$\longrightarrow \quad V(r,\bar{\phi},\lambda) = \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=-n}^{n} A_{nm}Y_{nm}(\bar{\phi},\lambda)$$

Spherical harmonic series

$$V(r,\bar{\phi},\lambda) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} A_{nm} Y_{nm}(\bar{\phi},\lambda)$$

$$Y_{nm}(\bar{\phi},\lambda) = \begin{cases} \cos m\lambda P_{n|m|}(\sin\bar{\phi}) & \text{for } m \le 0 \\ \sin m\lambda P_{nm}(\sin\bar{\phi}) & \text{for } m > 0 \end{cases}$$

→ **Surface** spherical harmonics of degree *n* and order *m*

$$\frac{1}{r^{n+1}} Y_{nm}(\bar{\phi},\lambda)$$

→ **Solid** spherical harmonics of degree *n* and order *m*

Legendre functions $P_{nm}(t)$

$$(1-t^2)\frac{d^2P_{nm}(t)}{dt^2} - 2t\frac{dP_{nm}(t)}{dt} + \left(n(n+1) - \frac{m^2}{1-t^2}\right) P_{nm}(t) = 0$$

$$P_{nm}(t) = \frac{1}{2^n\, n!}(1-t^2)^{\frac{m}{2}} \frac{d^{n+m}(t^2-1)^n}{dt^{n+m}}$$

→ **Legendre function** of degree *n* and order *m*

m=0

$$(1-t^2)\frac{d^2P_n(t)}{dt^2} - 2t\frac{dP_n(t)}{dt} + n(n+1)\ P_n(t) = 0$$

$$P_n(t) = \frac{1}{2^n\, n!}\frac{d^n(t^2-1)^n}{dt^n}$$

→ **Legendre polynomial** of degree *n*

→ $$P_{nm}(t) = (1-t^2)^{\frac{m}{2}} \frac{d^m\ P_n(t)}{dt^m}$$

→ $P_{n0}(t) = P_n(t)$

Legendre functions

n	m	$P_{nm}(t)$	f_{nm}	$\overline{P}_{nm}(t)$
0	0	1	1	1
1	0	t	$\sqrt{3}$	$\sqrt{3}\cdot t$
1	1	$\sqrt{1-t^2}$	$\sqrt{3}$	$\sqrt{3}\sqrt{1-t^2}$
2	0	$\frac{3}{2}t^2-\frac{1}{2}$	$\sqrt{5}$	$\sqrt{5}\cdot(\frac{3}{2}t^2-\frac{1}{2})$
2	1	$3t\sqrt{1-t^2}$	$\frac{1}{3}\sqrt{15}$	$\sqrt{15}t\sqrt{1-t^2}$
2	2	$3\ (1-t^2)$	$\frac{1}{6}\sqrt{15}$	$\frac{1}{2}\sqrt{15}(1-t^2)$
3	0	$\frac{5}{2}t^3-\frac{3}{2}t$	$\sqrt{7}$	$\sqrt{7}(\frac{5}{2}t^3-\frac{3}{2}t)$
3	1	$\sqrt{1-t^2}(\frac{15}{2}t^2-\frac{3}{2})$	$\sqrt{\frac{7}{6}}$	$\sqrt{\frac{7}{6}}\sqrt{1-t^2}(\frac{15}{2}t^2-\frac{3}{2})$
3	2	$15\ t(1-t^2)$	$\frac{1}{30}\sqrt{105}$	$\frac{1}{2}\sqrt{105}t(1-t^2)$
3	3	$15\ (1-t^2)^{\frac{3}{2}}$	$\frac{1}{4}\frac{1}{15}\sqrt{70}$	$\sqrt{\frac{35}{8}}(1-t^2)^{\frac{3}{2}}$

Recursive formulas of Legendre functions

$$P_n(t) = -\tfrac{n-1}{n}P_{n-2}(t) + \tfrac{2n-1}{n}t\ P_{n-1}(t) \qquad (n \geq 2;\ m=0)$$

$$P_{nm}(t) = -\tfrac{n+m-1}{n-m}P_{n-2,m}(t) + \tfrac{2n-1}{n-m}t\ P_{n-1,m}(t) \qquad (n \geq 3;\ 1 \leq m \leq n-2)$$

$$P_{n,n-1}(t) = (2n-1)t\ P_{n-1,n-1}(t) \qquad (n \geq 1;\ m=n-1)$$

$$P_{nn}(t) = (2n-1)\sqrt{1-t^2}P_{n-1,n-1}(t) \qquad (n \geq 2;\ m=n)$$

Normalization of surface spherical harmonics

$$\frac{1}{4\pi}\iint_\sigma Y_{nm}(\overline{\phi},\lambda)\, Y_{n'\,m'}(\overline{\phi},\lambda)\, d\sigma = \begin{cases} 0 & \text{if } n \neq n' \text{ and/or } m \neq m' \\ \frac{1}{2n+1} & \text{if } n = n' \text{ and } m = m' = 0 \\ \frac{1}{2(2n+1)}\frac{(n+m)!}{(n-m)!} & \text{if } n = n' \text{ and } m = m' \neq 0 \end{cases}$$

$$f_{nm} = \begin{cases} \sqrt{2n+1} & for\ m = 0 \\ \sqrt{2(2n+1)\frac{(n-m)!}{(n+m)!}} & for\ m \neq 0 \end{cases}$$

$$\overline{Y}_{nm}(\overline{\phi},\lambda) = f_{nm}\, Y_{nm}(\overline{\phi},\lambda)$$

$$\frac{1}{4\pi}\iint_\sigma \overline{Y}_{nm}(\overline{\phi},\lambda)\, \overline{Y}_{n'\,m'}(\overline{\phi},\lambda)\, d\sigma = \begin{cases} 0 & \text{if } n \neq n' \text{ and/or } m \neq m' \\ 1 & \text{if } n = n' \text{ and } m = m' \end{cases}$$

Orthogonality Principle of surface spherical harmonics

Normalized Legendre functions

$$\overline{P}_{nm}(t) = f_{nm}\, P_{nm}(t)$$

$$f_{nm} = \begin{cases} \sqrt{2n+1} & for\ m = 0 \\ \sqrt{2(2n+1)\frac{(n-m)!}{(n+m)!}} & for\ m \neq 0 \end{cases}$$

Recursive formulas of normalized Legendre functions

$$\bar{P}_n(t) = -\frac{\sqrt{2n+1}}{n}\frac{n-1}{\sqrt{2n-3}}\bar{P}_{n-2}(t) + \frac{\sqrt{2n+1}}{n}\sqrt{2n-1}\; t\bar{P}_{n-1}(t) \qquad (n \geq 2;\; m=0)$$

$$\bar{P}_{nm}(t) = -\sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}}\bar{P}_{n-2,m}(t) + \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}\; t\bar{P}_{n-1,m}(t) \qquad (n \geq 3;\; 1 \leq m \leq n-2)$$

$$\bar{P}_{n,n-1}(t) = \sqrt{2n+1} t\bar{P}_{n-1,n-1}(t) \qquad (n \geq 1;\; m = n-1)$$

$$\bar{P}_{nn}(t) = \sqrt{\frac{2n+1}{2n}}\sqrt{1-t^2}\bar{P}_{n-1,n-1}(t) \qquad (n \geq 2;\; m = n)$$

Normalized spherical harmonic series

$$V(r,\overline{\phi},\lambda) = \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=-n}^{n} A_{nm}Y_{nm}(\overline{\phi},\lambda)$$

$$\overline{Y}_{nm}(\overline{\phi},\lambda) = f_{nm}\, Y_{nm}(\overline{\phi},\lambda)$$

$$V(r,\overline{\phi},\lambda) = \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=-n}^{n} \overline{A}_{nm}\overline{Y}_{nm}(\overline{\phi},\lambda)$$

$$\overline{A}_{nm} = \frac{A_{nm}}{f_{nm}}$$

$$V(r,\overline{\phi},\lambda) = \sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\sum_{m=-n}^{n} \overline{A}'_{nm}\overline{Y}_{nm}(\overline{\phi},\lambda)$$

$$\overline{A}'_{nm} = \frac{\overline{A}_{nm}}{R^{n+1}}$$

Normalized spherical harmonic coefficients for a sphere of radius *R*

Harmonic analysis

$$V(r,\overline{\phi},\lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \overline{A}_{nm}\overline{Y}_{nm}(\overline{\phi},\lambda)$$

$$V(R,\overline{\phi},\lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \overline{A}_{nm}\overline{Y}_{nm}(\overline{\phi},\lambda)$$

$$\frac{1}{4\pi}\iint_\sigma V(R,\overline{\phi},\lambda)\overline{Y}_{ij}(\overline{\phi},\lambda)d\sigma = \frac{1}{4\pi}\iint_\sigma \left(\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \overline{A}_{nm}\overline{Y}_{nm}(\overline{\phi},\lambda)\right)\overline{Y}_{ij}(\overline{\phi},\lambda)d\sigma$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \overline{A}_{nm}\left(\frac{1}{4\pi}\iint_\sigma \overline{Y}_{nm}(\overline{\phi},\lambda)\overline{Y}_{ij}(\overline{\phi},\lambda)d\sigma\right) = \overline{A}_{ij}\cdot 1$$

$$\overline{A}_{nm} = \frac{1}{4\pi}\iint_\sigma V(R,\overline{\phi}',\lambda')\cdot\overline{Y}_{nm}(\overline{\phi}',\lambda')\cdot d\sigma(\overline{\phi}',\lambda')$$

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- Spherical harmonic series of earth's gravitational potential V
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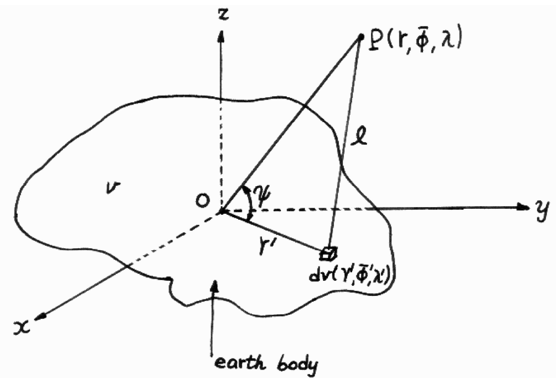
Gravitational potential of the earth



$$dV = \frac{G\rho dv}{\ell}$$

$$V = \int dV = G\iiint_v \frac{\rho dv}{\ell}$$

$$= G\iiint_v \frac{\rho(r', \bar{\phi}', \lambda') \cdot dv(r', \bar{\phi}', \lambda')}{\ell}$$

Coordinates of computation point ***P*** : (x, y, z) $(r, \bar{\phi}, \lambda)$

Coordinates of moving element dv : (x', y', z') $(r', \bar{\phi}', \lambda')$

Gravitational potential of the earth

$$\frac{1}{\ell} = \frac{1}{\sqrt{r^2 + r'^2 - 2r \cdot r' \cdot \cos\psi}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\psi)$$

$$P_n(\cos\psi) = \frac{1}{2n+1}\sum_{m=-n}^{n}\left(\overline{Y}_{nm}(\bar{\phi}, \lambda) \cdot \overline{Y}_{nm}(\bar{\phi}', \lambda')\right)$$

Addition theorem of normalized surface spherical harmonics

$$V(r, \bar{\phi}, \lambda) = G\iiint_v \frac{\rho(r', \bar{\phi}', \lambda') \cdot dv(r', \bar{\phi}', \lambda')}{\ell}$$

$$= \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=0}^{n}(\bar{a}_{nm}\cos m\lambda + \bar{b}_{nm}\sin m\lambda)\overline{P}_{nm}(\sin\bar{\phi})$$

Gravitational potential of the earth

$$V(r,\overline{\phi},\lambda)=\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=0}^{n}(\overline{a}_{nm}\cos m\lambda+\overline{b}_{nm}\sin m\lambda)\overline{P}_{nm}(\sin\overline{\phi})$$

$$\overline{a}_{nm}=\frac{G}{2n+1}\iiint_v (r')^n\cdot\cos m\lambda'\,\overline{P}_{nm}(\sin\overline{\phi}')\cdot\rho\,dv\quad(0\leq m\leq n)$$

$$\overline{b}_{nm}=\frac{G}{2n+1}\iiint_v (r')^n\cdot\sin m\lambda'\,\overline{P}_{nm}(\sin\overline{\phi}')\cdot\rho\,dv\quad(0< m\leq n)$$

Zero-degree term

For zero-degree term, $n = m = 0$

$$\overline{a}_{00}=G\iiint_v \rho dv=GM$$

$$V_{00}=\overline{a}_{00}\cdot\frac{1}{r}\overline{Y}_{00}(\overline{\phi},\lambda)=\frac{GM}{r}$$

→ The zero-degree term is the gravitational potential generated by a point mass equal to the total mass of the earth and placed at the origin of the coordinate system

Coordinates of the mass center of the earth

$$x_0 = \frac{1}{M} \iiint_v x' \cdot dm,$$

$$y_0 = \frac{1}{M} \iiint_v y' \cdot dm,$$

$$z_0 = \frac{1}{M} \iiint_v z' \cdot dm$$

→ $x_0 = y_0 = z_0 = 0$ **if** the origin O of the coordinate system is placed at the mass center of the earth

First-degree terms

For first-degree terms, $n = 1, m = 0, 1$

$$\bar{a}_{10} = \frac{G}{3} \iiint_v r' \cdot \overline{Y}_{1,0}(\overline{\phi}', \lambda') \cdot dm = \frac{G}{3} \iiint_v r' \cdot \sqrt{3} \sin \overline{\phi}' \cdot dm = \frac{GM}{\sqrt{3}} \cdot z_0$$

$$\bar{a}_{11} = \frac{G}{3} \iiint_v r' \cdot \overline{Y}_{1,-1}(\overline{\phi}', \lambda') \cdot dm = \frac{G}{3} \iiint_v r' \cdot \cos \lambda' \sqrt{3} \cos \overline{\phi}' \cdot dm = \frac{GM}{\sqrt{3}} \cdot x_0$$

$$\bar{b}_{11} = \frac{G}{3} \iiint_v r' \cdot \overline{Y}_{1,1}(\overline{\phi}', \lambda') \cdot dm = \frac{G}{3} \iiint_v r' \cdot \sin \lambda' \sqrt{3} \cos \overline{\phi}' \cdot dm = \frac{GM}{\sqrt{3}} \cdot y_0$$

→ All first-degree terms are zero **if** the origin O of the coordinate system is placed at the mass center of the earth

Moment of inertia of a rigid body

Moment of inertia of a body v with density ρ around axis ℓ :
(r is the distance from dm to the axis)

$$J_\ell = \iiint_v r^2\, dm = \iiint_v r^2\, \rho\, dv$$

Moment of inertia around x-, y- and z-axis, respectively:

$$J_x = \iiint_v (y^2 + z^2)\, dm,$$

$$J_y = \iiint_v (z^2 + x^2)\, dm,$$

$$J_z = \iiint_v (x^2 + y^2)\, dm$$

Product of inertia:

$$J_{xy} = \iiint_v xy\, dm,$$

$$J_{yz} = \iiint_v yz\, dm,$$

$$J_{xz} = \iiint_v zx\, dm$$

Inertia matrix of the earth

$$Q = \begin{bmatrix} J_x & J_{xy} & J_{xz} \\ J_{yx} & J_y & J_{yz} \\ J_{zx} & J_{zy} & J_z \end{bmatrix}$$

2nd-degree terms

For first-degree terms, *n = 2, m = 0, 1, 2*

$$\bar{a}_{20} = \frac{G}{\sqrt{5}}\left(\bar{J} - J_z\right), \quad \bar{J} = \frac{J_x + J_y}{2}$$

$$\bar{a}_{21} = \frac{G}{\sqrt{15}}\, J_{xz}$$

$$\bar{b}_{21} = \frac{G}{\sqrt{15}}\, J_{yz}$$

$$\bar{a}_{22} = \frac{\sqrt{15}}{10}\, G(J_y - J_x)$$

$$\bar{b}_{22} = \frac{\sqrt{15}}{5} G\, J_{xy}$$

$$f = \frac{a-b}{a} \approx \frac{1}{298.257}$$

$$f_m = \frac{J_z - \bar{J}}{J_z} \approx \frac{1}{305}$$

$\bar{a}_{20}$ → Ellipticity of the earth

$\bar{a}_{22}$ → Non-circular equator

Spherical harmonic series of V

$$V(r,\bar{\phi},\lambda) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} (\underline{\bar{a}_{nm}} \cos m\lambda + \underline{\bar{b}_{nm}} \sin m\lambda) \overline{P}_{nm}(\sin\bar{\phi})$$

$$\bar{a}_{nm} = \frac{G}{2n+1} \iiint_v (r')^n \cdot \cos m\lambda' \, \overline{P}_{nm}(\sin\bar{\phi}') \cdot \rho \, dv \quad (0 \le m \le n)$$

$$\bar{b}_{nm} = \frac{G}{2n+1} \iiint_v (r')^n \cdot \sin m\lambda' \, \overline{P}_{nm}(\sin\bar{\phi}') \cdot \rho \, dv \quad (0 < m \le n)$$

→ All coefficients have units !
All coefficients increase with ***n***

Spherical harmonic series of V

$$V(r,\overline{\phi},\lambda) = \frac{GM}{r} + \frac{GM}{a}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}(\overline{A}_{nm}\cos m\lambda + \overline{B}_{nm}\sin m\lambda)\overline{P}_{nm}(\sin\overline{\phi})$$

$$\overline{A}_{nm} = \frac{1}{a^n\cdot GM}\cdot\overline{a}_{nm}$$

$$\overline{B}_{nm} = \frac{1}{a^n\cdot GM}\cdot\overline{b}_{nm}$$

→

- ✓ Coefficients have no units !
- ✓ refer to a sphere of radius a and
- ✓ do not increase with n

$$V(r,\overline{\phi},\lambda) = \frac{GM}{r} + \frac{GM}{a}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=-n}^{n}\overline{A}_{nm}\overline{Y}_{nm}(\overline{\phi},\lambda)$$

↓

$$Y_{nm}(\overline{\phi},\lambda) = \begin{cases}\cos m\lambda P_{n|m|}(\sin\overline{\phi}) & \text{for } m \le 0\\ \sin m\lambda P_{nm}(\sin\overline{\phi}) & \text{for } m > 0\end{cases}$$

Spherical harmonic series of other quantities

Geoid height

$$N(r,\overline{\phi},\lambda) = \frac{T}{\gamma} = \frac{GM}{a\gamma}\sum_{n=2}^{\infty}(\frac{a}{r})^{n+1}\sum_{m=0}^{n}\{\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\}\,\overline{P}_{nm}(\sin\overline{\phi})$$

Gravity disturbance

$$\delta g(r,\overline{\phi},\lambda) = -\frac{\partial T}{\partial r} = \frac{GM}{a^2}\sum_{n=2}^{\infty}(n+1)(\frac{a}{r})^{n+2}\sum_{m=0}^{n}\{\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\}\,\overline{P}_{nm}(\sin\overline{\phi})$$

Gravity anomaly

$$\Delta g(r,\overline{\phi},\lambda) = -\frac{\partial T}{\partial r} - \frac{2}{r}T$$

$$= \frac{GM}{a^2}\sum_{n=2}^{\infty}(n-1)(\frac{a}{r})^{n+2}\sum_{m=0}^{n}\{\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\}\,\overline{P}_{nm}(\sin\overline{\phi})$$

Spherical harmonic series of the gravitational potential V' of the ellipsoid

$$V'(r,\overline{\phi},\lambda) = \frac{GM}{r}\left(1+\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n}\overline{J}_{2n}\,\bar{P}_{2n}(sin\overline{\phi})\right)$$

$$\overline{J}_{2n} = (-1)^n \frac{3\cdot e^{2n}\sqrt{4n+1}}{(2n+1)(2n+3)}\left(1-n+\frac{5n}{e^2}\cdot\overline{J}_2\right)$$

$$V'(r,\overline{\phi},\lambda) = \frac{GM}{r}\left(1+\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n}\sum_{m=0}^{n}(\overline{J}_{nm}cosm\lambda+\overline{K}_{nm}sinm\lambda)\bar{P}_{nm}(sin\overline{\phi})\right)$$

$\overline{J}_{nm} = 0$ if n is odd or $m \neq 0$

$\overline{J}_{nm} = \overline{J}_n$ if n is even and $m = 0$

$\overline{K}_{nm} = 0$ for all n, m

Spherical harmonic series of the disturbing potential T

$$T(r,\overline{\phi},\lambda) = W - U = V - V'$$

$$= \frac{GM}{a}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\left\{\overline{C}_{nm}\cos m\lambda+\overline{S}_{nm}\sin m\lambda\right\}\overline{P}_{nm}(\sin\overline{\phi})$$

$\overline{C}_{nm} = \overline{A}_{nm} - \overline{J}_{nm}$ $(n \geq 2,\ 0 \leq m \leq n)$

$\overline{S}_{nm} = \overline{B}_{nm}$ $(n \geq 2,\ 0 \leq m \leq n)$

$$T(r,\overline{\phi},\lambda) = \frac{GM}{a}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=-n}^{n}\bar{C}_{nm}\bar{Y}_{nm}(\bar{\phi},\lambda)$$

$\overline{C}_{nm} = \overline{A}_{nm} - \overline{J}_{nm}$ $(n \geq 2,\ -n \leq m \leq 0)$

$\overline{C}_{nm} = \overline{B}_{nm}$ $(n \geq 2,\ \ 0 < m \leq n)$

Estimation of potential coefficients

- Traditionally through analysis of perturbations of satellite orbits (low-degree coffefficients)
- Integration of surface gravity anomalies from gravity measurements on land and altimetry measurements over the oceans
- Combination of above two types information
- Dedicated satellite gravity missions: satellite-to-satellite tracking (GRACE), gradiometry measurements (GOCE)

Estimation using gravity anomalies

$$\Delta g(r,\bar{\phi},\lambda) = \frac{GM}{a^2}\sum_{n=2}^{\infty}(n-1)(\frac{a}{r})^{n+2}\sum_{m=0}^{n}\{\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\}\,\overline{P}$$

$r = a$

$$\Delta g(a,\bar{\phi},\lambda) = \frac{GM}{a^2}\sum_{n=2}^{\infty}(n-1)\sum_{m=-n}^{n}\overline{C}_{nm}\bar{Y}_{nm}(\bar{\phi},\lambda)$$

$$\frac{1}{4\pi}\iint_{\sigma}\Delta g(a,\bar{\phi},\lambda)\bar{Y}_{ij}(\bar{\phi},\lambda)d\sigma = \frac{1}{4\pi}\iint_{\sigma}\left(\frac{GM}{a^2}\sum_{n=2}^{\infty}(n-1)\sum_{m=-n}^{n}\overline{C}_{nm}\bar{Y}_{nm}(\bar{\phi},\lambda)\bar{Y}_{ij}(\bar{\phi},\lambda)\right)d\sigma$$

$(n-1)$

$$= \frac{GM}{a^2}\left(\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\overline{C}_{nm}\frac{1}{4\pi}\iint_{\sigma}\bar{Y}_{nm}(\bar{\phi},\lambda)\bar{Y}_{ij}(\bar{\phi},\lambda)d\sigma\right) = \frac{GM}{a^2}\cdot\left(\,(i-1)\,\overline{C}_{ij}\cdot 1\right)$$

Estimation using gravity anomalies

$$\bar{C}_{ij} = \frac{a^2}{4\pi GM(i-1)} \iint_\sigma \left(\Delta g(a,\bar{\phi},\lambda)\bar{Y}_{ij}(\bar{\phi},\lambda)\right) d\sigma(\bar{\phi},\lambda)$$

$$\bar{C}_{nm} = \frac{a^2}{4\pi GM(n-1)} \iint_\sigma \Delta g(a,\bar{\phi}',\lambda')\, \bar{Y}_{nm}(\bar{\phi}',\lambda')\, d\sigma(\bar{\phi}',\lambda')$$

$$\bar{C}_{nm} = \frac{a^2}{4\pi GM(n+1)} \iint_\sigma \delta g(a,\bar{\phi}',\lambda')\, \bar{Y}_{nm}(\bar{\phi}',\lambda')\, d\sigma(\bar{\phi}',\lambda')$$

Global Gravitational Models (GGM)

- a set of normalized spherical harmonic coefficients of the gravitation potential V of the earth, complete to degree and order n_{max}

  $\overline{A}_{nm}$ $\overline{B}_{nm}$ $(n \leq n_{max})$

- together with some ellipsoidal parameters: a, GM

- estimated from satellite measurements or ground measurements or a combination of them.

How many coefficients ???

$$\sum_{n=2}^{n_{\max}} (2n+1) = (n_{\max}-1)(n_{\max}+3) = n_{\max}^2 + 2\, n_{\max} - 3$$

If n_{max} = 2191, there will be more than

4.8 million potential coefficients !!!

Existing GGM

Name	$n_{\max}$	*Authors*	*Year*	*Remarks*
GEM 9	20	GSFC	1977	satellite data only
GEM 10C	180	GSFC	1978	combination solution
GEM T1	36	GSFC	1988	satellite data only
GEM T2	36	GSFC	1990	satellite data only
GEM T3	50	GSFC	1993	satellite data only
JGM 3	70	GSFC, UTA, OSU, CNES	1994	combination solution
Rapp 81	180	Rapp, OSU	1981	combination solution
OSU86C	250	Rapp, OSU	1986	combination solution
OSU89A	360	Rapp, OSU	1989	combination solution
OSU91A	360	Rapp, OSU	1991	combination solution
WGS84	42	DoD, USA	1984	for GPS ephemeris
GFZ93A	360	Gruber, GFZ, Potsdam	1993	
WDM94	360	Ning, WTUSM	1994	combination, with Chinese data
EGM96	360	NASA, NIMA, OSU, etc	1996	combination solution
GPM98C	1800	Wenzel, Hannover	1998	combination solution
EIGEN-3p	65	GFZ	2003	using 3-years CHAMP data
EIGEN-GRACE01S	120	GFZ	2003	using 39-days GRACE data
GGM01S	120	CSR	2003	using 111-days GRACE data
EGM 2008	2190	NGA, USA	2008	combination data

Global Gravitational Model GEM-T2

2	0	.15548961E-08	.00000000E+00
2	1	.00000000E+00	.00000000E+00
2	2	.24390067E-05	-.14000870E-05
3	0	.95703311E-06	.00000000E+00
3	1	.20307524E-05	.24960266E-06
3	2	.90353915E-06	-.61898576E-06
3	3	.72150732E-06	.14137252E-05
4	0	-.25039630E-06	.00000000E+00
4	1	-.53525575E-06	-.47413322E-06
4	2	.34825956E-06	.66402365E-06
4	3	.99131081E-06	-.20142885E-06
4	4	-.18936775E-06	.30896802E-06
5	0	.68688331E-07	.00000000E+00
5	1	-.60759526E-07	-.95025830E-07
[illegible]	[illegible]	[illegible]	[illegible]

GEM T2 coefficients for T
complete to degree/order 36

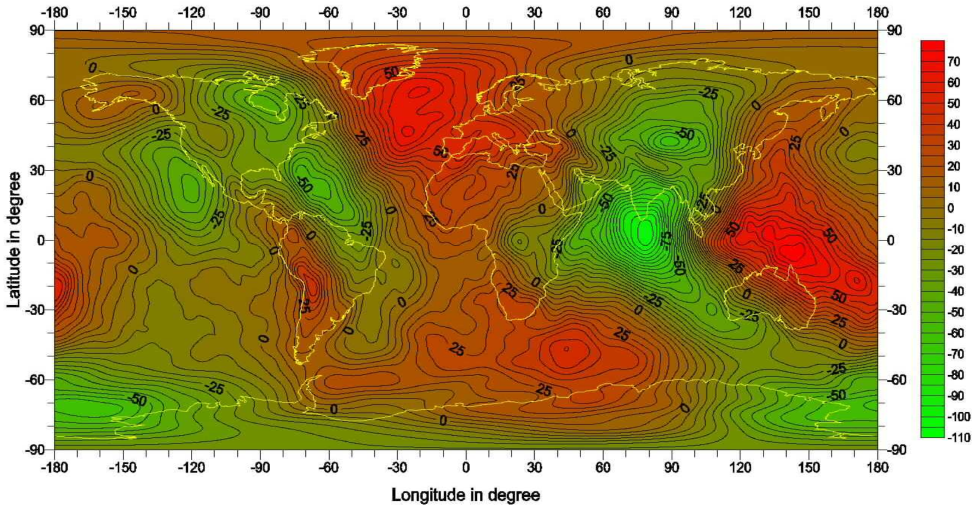
FORMAT(2I3,2E15.8)

File name: GEMT2.T

Global Gravitational Model GEM-T2

36	17	.29224034E-08	-.13638775E-08
36	18	.30582160E-10	.16655299E-09
36	19	-.69952820E-09	.77208428E-09
36	20	-.19394269E-08	-.20748737E-09
36	21	.23105800E-08	-.43550635E-08
36	22	-.45541772E-09	.16395183E-08
36	23	-.20414901E-08	-.15927978E-08
36	24	.34765079E-09	-.29341497E-08
36	25	.91178048E-09	.90987939E-08
36	26	.30418097E-08	.64559344E-08
36	27	-.84166488E-08	.53536590E-08
36	28	.14082796E-08	-.17119633E-08
36	29	.13412224E-08	-.91713750E-09
36	30	-.64072900E-08	.12875954E-08
36	31	-.21613715E-08	-.87760559E-09
36	32	.19452455E-09	.28940299E-08
36	33	-.40936686E-08	-.58626914E-08
36	34	-.21684159E-08	.18188953E-08
36	35	.46852355E-09	-.22694319E-08
36	36	.38624235E-09	.36811811E-09

World geoid map derived from JGM2S

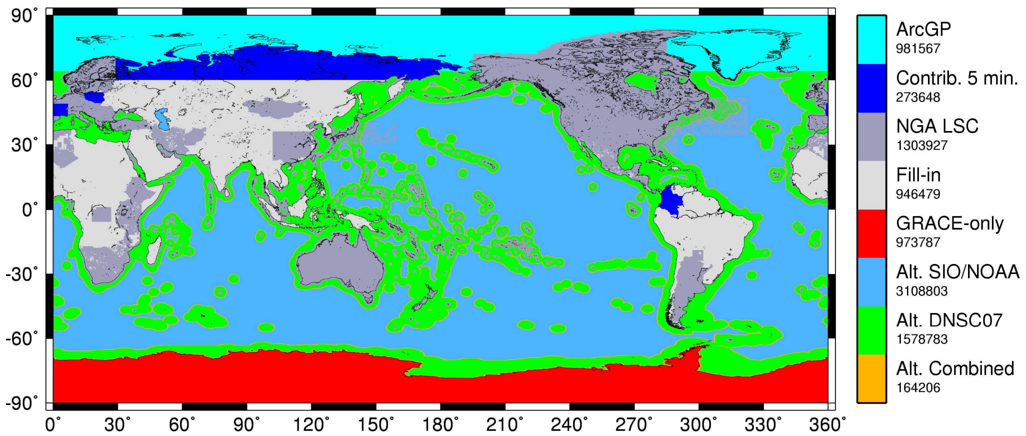


High degree combined model: EGM2008

- Developped by US National Geospatial-Intelligence Agency
- Use global 5’ by 5’ gravity anomaly data set
- Adjusted to / Combined with ITG-GRACE03S, to degree 180
- Final model complete to degree N_{max} **= 2190**
- GM-value not compatible with GRS-80
- Two versions: Non-tidal and zero-tidal versions
- Corresponding to height anomaly.

  Correction to geoid height is available.

Gravity data used to develop EGM2008



Pavlis et.al. (2008)

Gravity data used to develop EGM2008

Source	% Area	Min.	Max.	RMS	RMS σ
ArcGP	3.0	-192.0	281.8	30.2	3.0
Altimetry	63.2	-361.8	351.1	28.4	3.0
Terrestrial	17.6	-351.9	868.4	41.2	2.8
Fill-in	16.2	-333.0	593.5	46.8	7.6
Non Fill-in	83.8	-361.8	868.4	31.6	2.9
All	100.0	-361.8	868.4	34.5	4.1
(φ, λ)		19.4°, 293.5°	10.8°, 286.3°		

Statistics refer to <u>edited</u> and <u>downward continued</u> gravity anomalies.

The area void of high quality 5´ data (~16% of the globe) is also the "roughest" area of the gravity anomaly field. Pavlis et.al. (2008)

More recent GGMs vs GPS/levelling

Nr	Model ⇅	Nmax ⇅	USA ⇅ 6169 points	Canada ⇅ 2691 points	Europe ⇅ 1235 points	Australia ⇅ 201 points	Japan ⇅ 816 points	Brazil ⇅ 1112 points	All ▲ 12224 points
134	EIGEN-6C4	2190	0.247 m	0.126 m	0.210 m	0.212 m	0.079 m	0.446 m	0.2408 m
125	EIGEN-6C3STAT	1949	0.247 m	0.129 m	0.212 m	0.213 m	0.078 m	0.447 m	0.2415 m
117	EIGEN-6C2	1949	0.249 m	0.129 m	0.212 m	0.214 m	0.080 m	0.445 m	0.2423 m
112	EIGEN-6C	1420	0.247 m	0.136 m	0.214 m	0.219 m	0.082 m	0.448 m	0.2429 m
91	EGM2008	2190	0.248 m	0.128 m	0.208 m	0.217 m	0.083 m	0.460 m	0.2439 m
111	GIF48	360	0.319 m	0.209 m	0.275 m	0.236 m	0.275 m	0.474 m	0.3082 m
100	EIGEN-51C	359	0.335 m	0.234 m	0.289 m	0.234 m	0.312 m	0.476 m	0.3242 m
99	EIGEN-5C	360	0.341 m	0.278 m	0.303 m	0.244 m	0.339 m	0.524 m	0.3444 m
86	EIGEN-GL04C	360	0.339 m	0.282 m	0.336 m	0.244 m	0.321 m	0.541 m	0.3484 m
94	GGM03C	360	0.347 m	0.337 m	0.334 m	0.259 m	0.316 m	0.513 m	0.3588 m
81	EIGEN-CG01C	360	0.351 m	0.335 m	0.370 m	0.263 m	0.351 m	0.543 m	0.3700 m
84	EIGEN-CG03C	360	0.346 m	0.373 m	0.355 m	0.260 m	0.326 m	0.534 m	0.3714 m
131	GO_CONS_GCF_2_TIM_R5	280	0.398 m	0.310 m	0.371 m	0.336 m	0.450 m	0.505 m	0.3919 m
130	GO_CONS_GCF_2_DIR_R5	300	0.405 m	0.299 m	0.373 m	0.327 m	0.447 m	0.507 m	0.3937 m
118	GO_CONS_GCF_2_DIR_R4	260	0.404 m	0.322 m	0.393 m	0.337 m	0.476 m	0.512 m	0.4020 m
127	EIGEN-6S2	260	0.405 m	0.322 m	0.393 m	0.337 m	0.476 m	0.512 m	0.4025 m

Barthelmes, et.al (2015)