Introduction to Physical Geodesy

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Outline

- What and why ?
- Basic concepts
  - ✓ *Gravitation, rotation, gravity, potential functions*
  - ✓ *g-measurements: principles, absolute/relative*
  - ✓ *Gravity field, normal field, anomalous field*
- Stokes' classic theory
  - ✓ *Geodetic boundary value problems*
  - ✓ *Stokes' formula, Vening-Meinesz' formula*
- Non-classic theories
  - ✓ *Gravity reduction. Molodenskii'. Bjerhammar. Collocation*
  - ✓ *GGMs, modification of Stokes' formulas, satellite gravimetry*

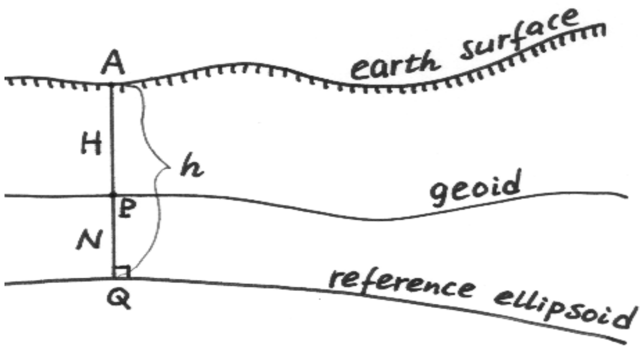
Physical Geodesy

- Measure the magnitude of the earth's gravity acceleration on the ground → ***gravimetry***
- Determine the **gravity field** of the earth by different types of measurements
- One of the three scientific tasks of geodesy
  - ✓*Figure of the earth*
  - ✓*Gravity field of the earth*
  - ✓*Geodynamic changes*
- Use gravity field information for geodetic purposes and for other scientific research

Why gravity field ?

- To define geodetic reference systems, in particular the geoid and the height systems
- To convert between GNSS-derived ellipsoidla heights and levelled heights above the sea level

Ellipsoidal height vs orthometric height



h=H+N

h: ellipsoidal height

H: orthometric height (height above *Mean Sea Level*)

N: geoidal height

Why gravity field ?

- To define geodetic reference systems, in particular the geoid and height systems
- For convesion between GNSS-derived ellipsoidla heights and levelled heights above the sea level
- For determination of precise orbits of artificial satellites
- For studies of the earth system: the *crust*, the *mantle*, the *oceans* and the *atmosphere* $\longrightarrow$ *global changes*

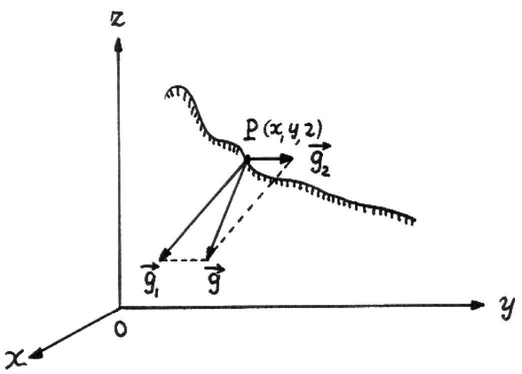
Basic equations of ocean circulation

$$\begin{cases} \frac{\partial p}{\partial h} = g \cdot \rho\,(\phi, \lambda, t) \\ \frac{\partial p}{a\,\partial \phi} = \frac{g}{a} \frac{\partial h(\phi,\lambda,t)}{\partial \phi} \\ \frac{\partial p}{a\,\cos\phi\,\partial \lambda} = \frac{g}{a\,\cos\phi} \frac{\partial h(\phi,\lambda,t)}{\partial \lambda} \end{cases}$$

- $p,\ g,\ \rho$ denote pressure, local gravity and ocean density, respectively.
- ϕ, λ denote latitude and longitude and
- h denotes the sea surface elevation above the geoid.

$\longrightarrow$ The geoid is needed to model global ocean circulations

Gravitation, rotation & gravity



$\vec{g_1}$	Gravitational force
$\vec{g_2}$	Centrifugal force or: rotational force
$\vec{g}$	Gravity force

$$\vec{g} = \vec{g_1} + \vec{g_2}$$

$g = |\vec{g}|$ $\longrightarrow$ **to be measured !**

Units of gravity (acceleration)

$g = |\vec{g}|$

$\sim 9.8\ m/s^2$

present accuracy

$g \sim 9.81\ 234\ 567\ m/s^2$

$1\ Gal = 1\ cm/s^2 = 1000\ mGal = 1000\ 000\ \mu Gal$

$\rightarrow$ $g \sim 980\ Gal$

$\sim 980\ 000\ mGal$

$\sim 980\ 000\ 000\ \mu Gal$

$\sim 10^9\ \mu Gal \rightarrow ppb$

Force vectors vs potential functions

a force vector is the gradient of the potential function

$$\vec{g} = grad(W) = \begin{bmatrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \\ \frac{\partial W}{\partial z} \end{bmatrix} \qquad g = |\vec{g}| = -\frac{dW}{dn}$$

$$g = |\vec{g}| = \sqrt{\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2}$$

$$\vec{g} = \vec{g_1} + \vec{g_2} \qquad W = V + \Omega$$

Gravitational potential (V)

Gravity potential or Geopotential (W)

Rotational potential (Ω)

Equipotential surfaces

$$W(x,y,z) = constant\ C$$

$\longrightarrow$ *an* equipotential function, *or: an equipotential surface*

***That* equipotential surface** which coincides with the Mean Sea Level (MSL) and extends under the continents is called the **geoid**

$$\frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz = \vec{g} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \equiv 0$$

$\downarrow$

Gravity vector is orthogonal to the equipotential surface !

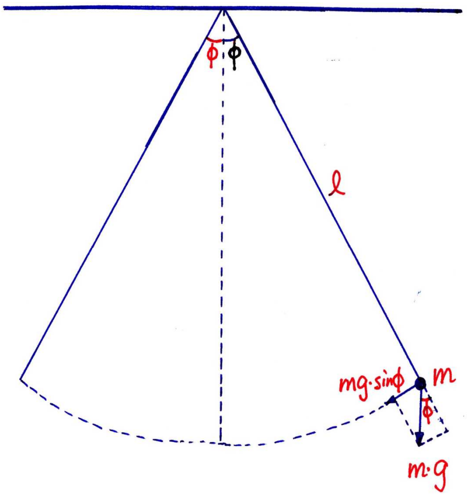
Trend in gravity variations

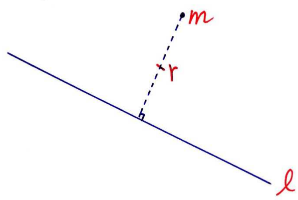
- Gravity changes from place to place
- Gravity increases from equator toward Poles
  - longer distance to geocenter at equator
  - centrifual force decreases
  - negative effect of centrifugal force decreases
- Gravity varies irregularly *locally* due to mass variations
- Other factors influencing gravity
  - *earthquake, crustal motion, mass movement inside*
  - *effect of the Sun and Moon. Sea level change.*

Gravity measurements

- Types of measurements
  - ✓ Absolute gravity measurements: g ~9.8
  - ✓ Relative gravity measurements: g_2-g_1
- Types of methods
  - ✓ Dynamic methods: observing motions
  - ✓ Static methods: observing a state of equilibrium
- Practical implementation
  - ✓ National gravity network (→ reference system), measured by **absolute** and relative gravimeters
  - ✓ Detail gravity survey, using relative gravimeters

Measuring gravity using a pendulum





Moment of inertia of point mass m wrt axis ℓ

$$J_\ell = m\, r^2$$

Rotational equation of a pendulum

$$M_\ell = J_\ell \cdot \frac{d\omega}{dt}$$

$$F = m \cdot \frac{dv}{dt}$$

$$F = m \cdot g \cdot \sin\phi$$

$$M_\ell = F \cdot \ell = -mg \cdot \ell \cdot \sin\phi$$

$$J_\ell = m\ell^2$$

$$J_\ell = \sum_{k=1}^{n} (r_k^2 \cdot m_k) \qquad J_\ell = \iiint_v r^2 \, dm = \iiint_v r^2 \, \rho \, dv$$

$$m\ell^2 \cdot \frac{d\omega}{dt} = -mg \cdot \ell \cdot \sin\phi$$

Equation of a pendulum

$$m\ell^2 \cdot \frac{d\omega}{dt} = -mg \cdot \ell \cdot \sin\phi$$

$\omega = \frac{d\phi}{dt}$ and $\sin\phi \approx \phi$ $\longrightarrow$ $\frac{d^2\phi}{dt} + \frac{g}{\ell} \cdot \phi = 0$

$$\phi(t) = A \cdot \sin\left(\sqrt{\frac{g}{\ell}}\, t + \phi_0\right)$$

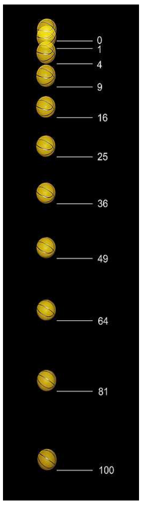
$$T = \frac{2\pi}{\sqrt{\frac{g}{\ell}}} = 2\pi\sqrt{\frac{\ell}{g}}$$

$\longrightarrow$ $\boxed{g = \frac{4\pi^2\, l}{T^2}}$

$\frac{m_T}{T} \approx 10^{-6}$, $\frac{m_\ell}{\ell} \approx 10^{-6}$ $\longrightarrow$ $\frac{m_g}{g} = \sqrt{\left(\frac{m_\ell}{\ell}\right)^2 + 4\left(\frac{m_T}{T}\right)^2} \approx 2.24 \cdot 10^{-6}$

(or $m_g \approx 2.2\, mGal$)

Absolute gravity measurement



Measure distance and time traveled by a falling body

(photography time unit =1/20 s, distance unit=12mm)

$$h = h_0 + v_0\, t + \frac{1}{2} g\, t^2$$

$h_1,\ h_2,\ h_3$ for three time epochs $t_1,\ t_2,\ t_3$

$$g = 2\frac{s_2/\Delta t_2 - s_1/\Delta t_1}{\Delta t_2 - \Delta t_1}$$

$$s_1 = h_2 - h_1, \quad s_2 = h_3 - h_1$$

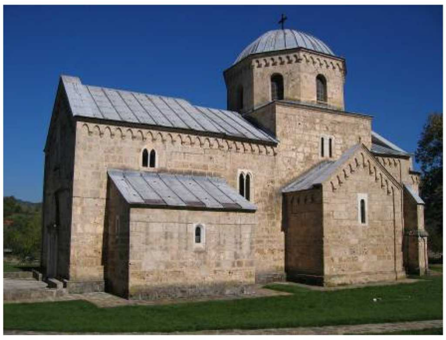
$$\Delta t_1 = t_2 - t_1, \quad \Delta t_2 = t_3 - t_1$$

Absolute gravimeter Lacoste-FG5

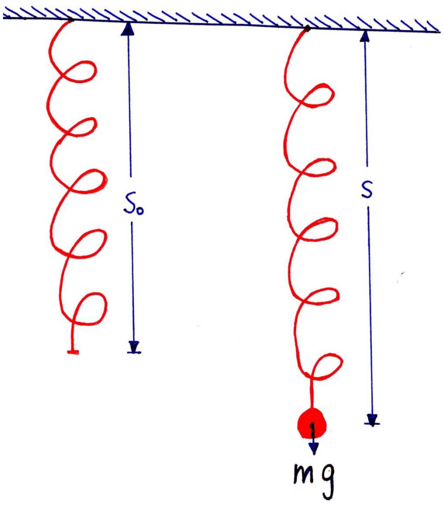
- Measure distance and time traveled by a falling body
- use laser interferometry method

- The instrument is run from a PC with special software ”g”
- Measure during at least 24 hours. 48 hours or more for best result
- Only for indoor measurements
- Demands a flat surface
- demounted after measuring and stored in large and heavy boxes.

Swedish FG5 in Gradac



Measuring gravity using a spring



$$k(s - s_0) = m \cdot g$$

$$g_2 - g_1 = \frac{k}{m}(s_2 - s_1)$$

Zero drift correction:

$$d = \frac{g_1 - g_0}{t_1 - t_0}$$

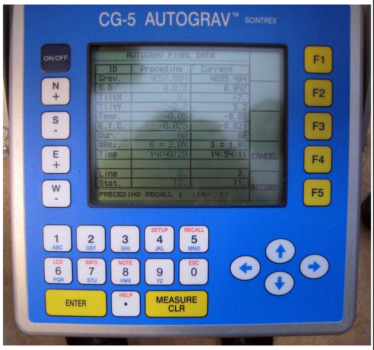
$$v = d \cdot (t - t_0)$$

Relative gravimeters

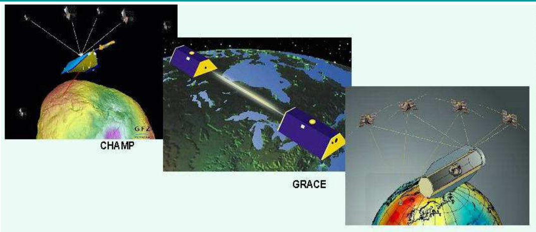
Worden

LaCost-Romberg

CG-5



Dedicated satellite gravity missions



- CHAMP – **CHA**llenging **M**inisatellite **P**ayload
  - ✓ HL-SST, German project launched in July 2000
- GRACE – **G**ravity **R**eceovery **A**nd **C**limate **E**xperiment
  - ✓ LL-SST, US-German project launched in March 2002
  - ✓ two satellites at 250 km altitude, about 200km away from each other
  - ✓ relative velocities measured with accuracy of about 0.001 mm/s
- GOCE – **G**ravity field and steady-state **O**cean **C**irculation **E**xplorer

"Variational approach" to determine W

$$W = V + \Omega$$

- If earth rotation velocity is known, Ω can be computed from positional coordinates. The classical task of physical geodesy is to determine V.
- From ground measurements g, it is very difficult or impossible to detemrine V.

- Create a theoretical gravity field very close to the real gravity of the earth. This theoretical field has gravitational potential U and gravity γ which can be computed.
- It is possible or easier to determine V- U from g- γ

Normal & anomalous gravity fields

- Definition of the normal gravity field
  - ✓ 4 assumptions to define the normal field
  - ✓ 4 defining/derived constants of GRS 80 (WGS 84)
  - ✓ Normal gravity formulas
- Anomalous gravity field
  - ✓ Quantities of the anomalous field: $T, N, \ldots\ldots$
  - ✓ Relationship between T and other quantities
  - ✓ Bruns' theorem Derive !
  - ✓ Gravimetric boundary value condition Derive !

Normal gravity field of the earth

- The total mass *M* of the reference ellipsoid (an ellipsoid of revolution) is equal to the total mass of the earth
- The ellipsoid rotates around its minor axis at same angular velocity ω as the earth
- Ellipsoidal surface is an equipotential surface with potential U_0 equal to the geopotential W_0 on the geoid
- Difference between polar/equatorial moment of inertia equal to that of the earth

Normal Gravity Field

- Computation of the normal gravity field
  - 4 defining constants needed: a, GM, J_2, ω (or: a, GM, f, ω)
  - Derived constants: $b, f, e, e', m, k, \ldots\ldots$
  - Normal gravity formulas. Mean normal gravity
- Existing normal gravity fields
  - GRS 1930 (a, f, γ_e, ω)
  - GRS 1980 (a, GM, J_2, ω)
  - WGS 84 (G873) (a, f, GM, ω)

Defining constants

Table 3.4: Defining constants of the Geodetic Reference System 1930

Notation	*Constant*	*Unit*	*Numerical value*
a	semi-major axis	m	6 378 388.000
f	flattening		1/297.000
γ_e	equatorial gravity	Gal	978.049 000
ω	angular velocity	s^{-1}	$0.729\,211\,51 \cdot 10^{-4}$

Table 3.5: Defining constants of the Geodetic Reference System 1980

Notation	*Constant*	*Unit*	*Numerical value*
a	semi-major axis	m	6 378 137.000
GM	Product of G and total mass M	$m^3 s^{-2}$	$0.398\,6005 \cdot 10^{15}$
J_2	dynamic form factor $\dfrac{C-A}{Ma^2}$		0.001 082 63
ω	angular velocity	s^{-1}	$0.729\,211\,51 \cdot 10^{-4}$

Derived constants of GRS 80

Notation	Constant	Unit	Value
b	semi-minor axis	*metre*	6 356 752.3141
f	geometrical flattening		0.003 352 810 681 1/298.257 222 101
e^2	first eccentricity squared		0.006 694 380 023
e'^2	second eccentricity squared	sec^{-1}	0.006 739 496 775
U_0	normal potential on the ellipsoid	$m^2 \cdot sec^{-2}$	62 636 860.850
γ_p	normal gravity on the Poles	Gal	983.218 636 85
γ_e	normal gravity on the equator	Gal	978.032 677 15
f^*	gravity flattening		0.005 302 440 112 1/188.592 417 552
k	$(b\gamma_p - a\gamma_e)/(a\gamma_e)$		0.001 931 851 353
m	$\omega^2 a^2 b/(GM)$		0.003 449 786 003 1/289.873 052 743
γ_{45}	normal gravity at latitude 45^0	Gal	980.619 920 3
$\bar{\gamma}$	global mean normal gravity	Gal	979.764 465 6

Normal potential U

$$x = \sqrt{u^2 + E^2}\cos\beta\cos\lambda$$
$$y = \sqrt{u^2 + E^2}\cos\beta\sin\lambda$$
$$z = u\sin\beta$$

$$E^2 = a^2 - b^2$$

$$U = V' + \Omega$$

$$\Omega = \frac{1}{2}\omega^2(x^2 + y^2) = \frac{1}{2}\omega^2(u^2 + E^2)\cos^2\beta$$

$$U(x,y,z) = U(u,\beta) = \frac{GM}{E}\tan^{-1}\frac{E}{u} + \frac{1}{2}\omega^2 a^2 \frac{q}{q_0}(\sin^2\beta - \frac{1}{3}) + \frac{1}{2}\omega^2(u^2 + E^2)\cos^2\beta$$

$$U_0 = U(b,\beta) = \frac{GM}{E}\tan^{-1}\frac{E}{b} + \frac{1}{3}\omega^2 a^2$$

$$q = \frac{1}{2}\left[\left(1 + 3\frac{u^2}{E^2}\right)\tan^{-1}\frac{E}{u} - \frac{1}{3}\frac{u}{E}\right]$$
$$q_0 = \frac{1}{2}\left[\left(1 + 3\frac{b^2}{E^2}\right)\tan^{-1}\frac{E}{b} - \frac{1}{3}\frac{b}{E}\right]$$

Normal gravity on the ellipsoid

$$\gamma = \sqrt{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2} = -\frac{dU}{dn'}$$

$$\gamma = \frac{a\gamma_p\sin^2\beta + b\gamma_e\cos^2\beta}{\sqrt{a^2\sin^2\beta + b^2\cos^2\beta}} = \frac{a\gamma_e\cos^2\phi + b\gamma_p\sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \gamma_e\frac{1 + k\sin^2\phi}{\sqrt{1 - e^2\sin^2\phi}}$$

$\gamma_e =$ normal gravity at the equator
$\gamma_p =$ normal gravity at the poles
$\beta =$ the reduced latitude
$\phi =$ the geodetic latitude
$k = (b\gamma_p - a\gamma_e)/(a\gamma_e)$

$$\gamma \approx \gamma_e(1 + f^*\sin^2\phi - \frac{1}{4}f_4\sin^2 2\phi)$$

Normal gravity above ellipsoid. Clairaut's theorem

$$\gamma_h = \gamma(\phi, h) = \gamma - \frac{2\gamma_e}{a}\left[1 + f + m + (-3f + \frac{5}{2}m)\sin^2\phi\right] h + \frac{3\gamma_e}{a^2} h^2$$

$$f^* = (\gamma_p - \gamma_e)/\gamma_e = f_2 + f_4$$
$$f_2 = -f + \tfrac{5}{2}m + \tfrac{1}{2}f^2 - \tfrac{26}{7}fm + \tfrac{15}{4}m^2$$
$$f_4 = -\tfrac{1}{2}f^2 + \tfrac{5}{2}fm$$
$$m = \omega^2 a^2 b/(GM)$$
$$f = (a-b)/a$$

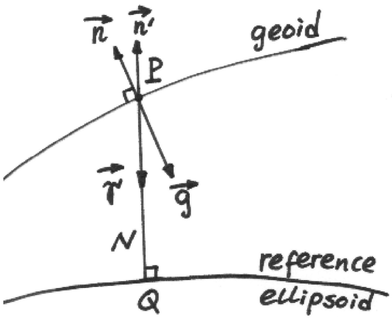
$$f + f^* = \frac{5}{2}m$$

$\longrightarrow$ Geometrical flattening *f* can be determined from measurements of a physical quantity, i.e. gravity.

Anomalous gravity field

- The **real** gravity field of the earth **minus** the **normal** gravity field of the reference ellipsoid is called the **anomalous** gravity field of the earth
- Anomalous gravity field can be described by different quantities
- All these quantities describe the differences between the **real** and the **normal** gravity field
- All these quantities are much smaller than the original quantities. Thus spherical approximations or linearizations are allowed

Anomalous quantities



- Disturbing potential T

  $T_p = W_p - U_p$

  $T_p = (V_p + \Omega_p) - (V'_p + \Omega_p) = V_p - V'_p$

- Gravity disturbance

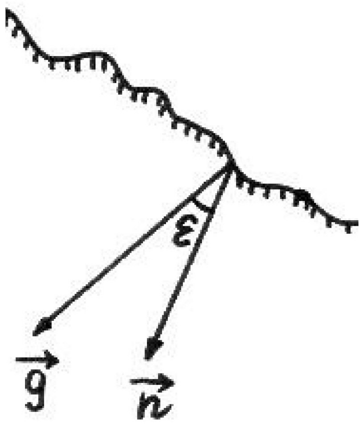
  $\delta g_p = g_p - \gamma_p$

- Gravity anomaly

  $\Delta g_p = g_p - \gamma_q$

- Geoid height N: from Q to P along ellipsoidal normal at Q

Deflection of the vertical



- Gravity vector

  $$\vec{g} = -g \begin{bmatrix} \cos\Phi\cos\Lambda \\ \cos\Phi\sin\Lambda \\ \sin\Phi \end{bmatrix}$$

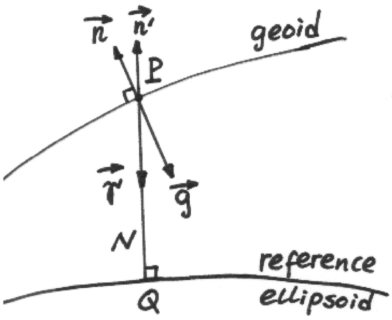
- Components of the deflection of the vertical

  $$\varepsilon = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

  $$\phi = \Phi - \xi$$
  $$\lambda = \Lambda - \eta / \cos\phi$$

  $$\varepsilon_\alpha = \xi\cos\alpha + \eta\sin\alpha$$

Relations between different quantities



$$U_p = U_q + \frac{\partial U}{\partial n'} N + \frac{1}{2!}\frac{\partial^2 U}{\partial n'^2} N^2 + \cdots$$

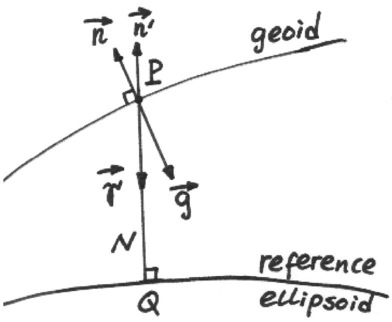
$$\approx U_q + \frac{\partial U}{\partial n'} N = U_q - \gamma_q \cdot N$$

$$T_p = W_p - U_p = U_q - U_p$$

$$= U_q - (U_q - \gamma_q \cdot N) = \gamma_q \cdot N$$

Bruns' theorem $\longrightarrow$ $$N = \frac{T_p}{\gamma_q}$$

Relations between different quantities

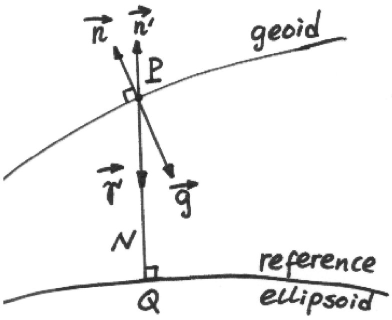


$$\delta g_p = g_p - \gamma_p = \left(-\frac{\partial W_p}{\partial n}\right) - \left(-\frac{\partial U_p}{\partial n'}\right)$$

$$\approx \left(-\frac{\partial W_p}{\partial r}\right) - \left(-\frac{\partial U_p}{\partial r}\right) = -\frac{\partial (W_p - U_p)}{\partial r}$$

$\longrightarrow$ $$\delta g_p = -\frac{\partial T_p}{\partial r}$$

Relations between different quantities



$$\Delta g_p = g_p - \gamma_q = \left(-\frac{\partial W_p}{\partial n}\right) - \left(-\frac{\partial U_q}{\partial n'}\right)$$

$$\approx \left(-\frac{\partial (T_p + U_p)}{\partial r}\right) - \left(-\frac{\partial U_q}{\partial r}\right)$$

$$= -\frac{\partial T_p}{\partial r} - \frac{\partial}{\partial r}\{U_q - \gamma_q \cdot N\} + \frac{\partial U_q}{\partial r}$$

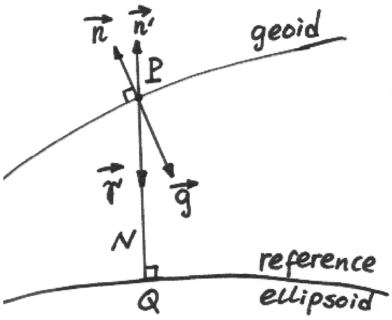
$$= -\frac{\partial T_p}{\partial r} + \frac{\partial \gamma_q}{\partial r} \cdot N$$

$$\frac{\partial \gamma_q}{\partial r} \approx \frac{\partial}{\partial r}\left(\frac{GM}{r^2}\right) = -\frac{2}{r}\frac{GM}{r^2} \approx -\frac{2}{r}\gamma_q$$

Gravimetric boundary value condition →

$$\Delta g_p = -\frac{\partial T_p}{\partial r} - \frac{2}{r} T_p$$

Relations between different quantities



$$\varepsilon \approx -\frac{\Delta N}{\Delta s}$$

$$\varepsilon = -\frac{dN}{ds}$$

$$\begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{dN}{dx} \\ -\frac{dN}{dy} \end{pmatrix}$$

$$= -\begin{pmatrix} \frac{dN}{R d\phi} \\ \frac{dN}{R \cos\phi \, d\lambda} \end{pmatrix} = -\frac{1}{\gamma_q}\begin{pmatrix} \frac{dT}{R d\phi} \\ \frac{dT}{R \cos\phi \, d\lambda} \end{pmatrix}$$

Basic concepts: conclusion

If we can determine the disturbing potential T,

all other quantities can be derived from T

Laplace's equation

- in rectangular coordinates (x,y,z)

$$\Delta(V) \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- in spherical coordinate $(r, \overline{\phi}, \lambda)$
:

$$r^2 \frac{\partial^2 V}{\partial r^2} + 2r \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \overline{\phi}^2} - \tan \overline{\phi} \frac{\partial V}{\partial \overline{\phi}} + \frac{1}{\cos^2 \overline{\phi}} \frac{\partial^2 V}{\partial \lambda^2} = 0$$

V is called a **harmonic function**

Example – Reciprocal distance

$$\ell = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

$$\frac{\partial \ell}{\partial x} = \frac{2(x-x')}{2\ \ell} = \frac{x-x'}{\ell}$$

$$f(x,y,z) = \frac{1}{\ell}$$

$$\left(\frac{u}{v}\right)' = \frac{u'\ v - u\ v'}{v^2}$$

$$\frac{\partial}{\partial x}(\frac{1}{\ell}) = -\frac{1}{\ell^2}\frac{\partial \ell}{\partial x} = -\frac{x-x'}{\ell^3}$$

$$\frac{\partial^2}{\partial x^2}(\frac{1}{\ell}) = \frac{\partial}{\partial x}\left(-\frac{x-x'}{\ell^3}\right) = -\frac{1\cdot\ell^3-(x-x')\ 3\ell^2\ \frac{x-x'}{\ell}}{\ell^6} = \frac{\ell^2-3\cdot(x-x')^2}{\ell^5}$$

Reciprocal distance

$$\frac{\partial}{\partial y^2}(\frac{1}{\ell}) = \frac{\ell^2-3\cdot(y-y')^2}{\ell^5}$$

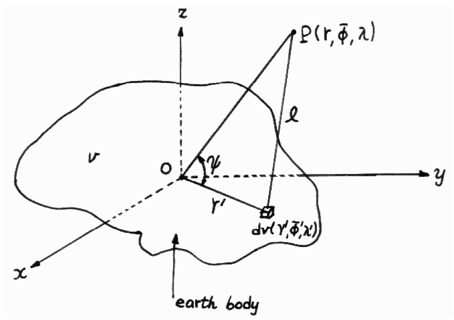
$$\frac{\partial}{\partial z^2}(\frac{1}{\ell}) = \frac{\ell^2-3\cdot(z-z')^2}{\ell^5}$$

$$\Delta(\frac{1}{\ell}) = \frac{\partial^2}{\partial x^2}(\frac{1}{\ell}) + \frac{\partial^2}{\partial y^2}(\frac{1}{\ell}) + \frac{\partial^2}{\partial z^2}(\frac{1}{\ell})$$

$$= \frac{3\cdot\ell^2-3\cdot\left\{(x-x')^2+(y-y')^2+(z-z')^2\right\}}{\ell^5} = \frac{3\ell^2-3\ell^2}{\ell^5} = 0$$

$\rightarrow$ *f(x,y,z)* is a harmonic function

Gravitational potential of a body



$$\ell = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\Delta(\frac{1}{\ell}) = 0$$

$$\Delta(V) = \Delta\left(G \iiint_v \frac{\rho dv}{\ell}\right) =$$

$$= G \iiint_v \Delta(\frac{1}{\ell})\, \rho dv = 0$$

→ The external gravitational potential V is always a harmonic function

BVPs in potential theory

V_p	=	potential function at point P
S	=	surface of the attracting body
v	=	external space outside the attracting body
n	=	normal of S
f_p	=	a given function of point P residing on S
$\alpha,\ \beta$	=	two given functions

1st boundary value problem – Dirichlet's problem

given: $\Delta(V_p) = 0$	for $P \in v$
given: $V_p = f_p$	for $P \in S$
sought: V_p	for $P \in v$

BVPs in potential theory & geodesy

2nd boundary value problem – Neumann's problem

given:	$\Delta(V_p) = 0$	for $P \in v$
given:	$\frac{\partial V_p}{\partial n} = f_p$	for $P \in S$
sought:	V_p	for $P \in v + S$

3rd boundary value problem – mixed problem

given:	$\Delta(V_p) = 0$	for $P \in v$
given:	$\alpha V_p + \beta \frac{\partial V_p}{\partial n} = f_p$	for $P \in S$
sought:	V_p	for $P \in v + S$

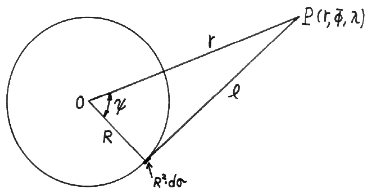
3rd **geodetic** boundary value problem – gravmetric BVP

Given :	$\Delta(T) = 0$	in v
Given :	$-\frac{\partial T}{\partial r} - \frac{2}{r} T = \Delta g$	on S
Sought :	$T = ?$	on S and in v

(Extended) Stokes's formula

$$T(r,\phi,\lambda) = \frac{R}{4\pi} \iint_\sigma S(r,\psi) \cdot \Delta g(R,\phi',\lambda') \cdot d\sigma(\phi',\lambda')$$

$$S(r,\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi)$$



$$S(r,\psi) = 2\frac{R}{\ell} + \frac{R}{r} - 3\frac{R\ell}{r^2} - 5\frac{R^2}{r^2}\cos\psi - 3\frac{R^2}{r^2}\cos\psi \ln \frac{r - R\cos\psi + \ell}{2r}$$

$$\ell = \sqrt{r^2 + R^2 - 2rR\cos\psi}$$

Stokes's formula

r = R

$$T(R,\phi,\lambda) = \frac{R}{4\pi}\iint_\sigma S(\psi)\cdot\Delta g(R,\phi',\lambda')\cdot d\sigma(\phi',\lambda')$$

$$N = \frac{T(R,\phi,\lambda)}{\gamma} = \frac{R}{4\pi\gamma}\iint_\sigma S(\psi)\cdot\Delta g(R,\phi',\lambda')\cdot d\sigma(\phi',\lambda')$$

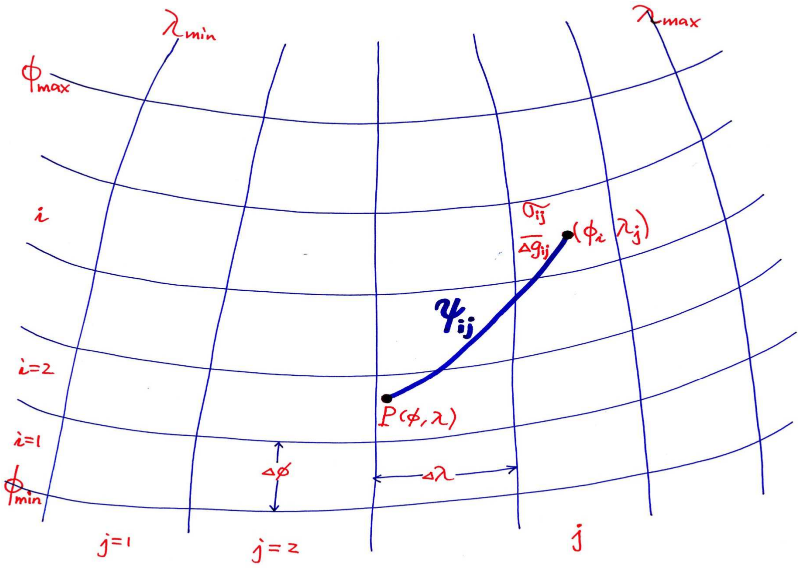
Stokes's formula

Stokes's function:

$$S(\psi) = [S(r,\psi)]_{r\to R} = \sum_{n=2}^{\infty}\frac{2n+1}{n-1}P_n(\cos\psi)$$

$$S(\psi) = \frac{1}{\sin\frac{\psi}{2}} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi\ln(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2})$$

$$S(t) = \sqrt{\frac{2}{1-t}} - 6\sqrt{\frac{1-t}{2}} + 1 - 5t - 3t\ln\left(\sqrt{\frac{1-t}{2}} + \frac{1-t}{2}\right)$$



$$\phi_i = \phi_{min} + (i-\tfrac{1}{2})\Delta\phi$$
$$\lambda_j = \lambda_{min} + (j-\tfrac{1}{2})\Delta\lambda$$
$$A_{ij} = \iint_{\sigma_{ij}} d\sigma = 2\cdot\Delta\lambda\cdot\sin\tfrac{\Delta\phi}{2}\cos\phi_i$$

Numerical integration

R = mean earth radius ($\approx$ 6371 km)

$cos\psi_{ij} = \sin\phi \ \sin\phi_i + \cos\phi \ \cos\phi_i \cos(\lambda - \lambda_j)$
$\phi_i = \phi_{min} + (i - \frac{1}{2})\Delta\phi$
$\lambda_j = \lambda_{min} + (j - \frac{1}{2})\Delta\lambda$
$A_{ij} = \iint_{\sigma_{ij}} d\sigma = 2 \cdot \Delta\lambda \cdot \sin\frac{\Delta\phi}{2}\cos\phi_i$

γ = normal gravity on the reference ellipsoid

$\Delta\bar{g}_{ij}$ = mean gravity anomaly for block σ_{ij}

ψ_{ij} = spherical distance from the computation point (ϕ, λ) to the block centre of σ_{ij}

ϕ_{min}, λ_{min} = the minimum latitude and minimum longitude of the integration area

$\Delta\phi$, $\Delta\lambda$ = block sizes, i.e. the latitude/longitude differences of a block

A_{ij}= area of block σ_{ij} .

Truncation & numerical integration

$$N = \frac{T(R,\phi,\lambda)}{\gamma} = \frac{R}{4\pi\gamma}\iint_{\sigma} S(\psi)\cdot\Delta g(R,\phi',\lambda')\cdot d\sigma(\phi',\lambda')$$

$$\hat{N} = \frac{R}{4\pi\gamma}\iint_{\sigma_0} \Delta g \ S(\psi) d\sigma = \frac{R}{4\pi\gamma}\sum_i\sum_j\iint_{\sigma_{ij}} \Delta g S(\psi)\cdot d\sigma_{ij}$$

$$\approx \frac{R}{4\pi\gamma}\sum_i\sum_j\iint_{\sigma_{ij}} \Delta\bar{g}_{ij} S(\psi_{ij})\cdot d\sigma_{ij} = \frac{R}{4\pi\gamma}\sum_i\sum_j \Delta\bar{g}_{ij} S(\psi_{ij})\iint_{\sigma_{ij}} d\sigma_{ij}$$

$$= \frac{R}{4\pi\gamma}\sum_i\sum_j\{\Delta\bar{g}_{ij} S(\psi_{ij}) A_{ij}\}$$

Gravity anomaly data file

```
 1  1   59   3 15  5   -15.24
 1  2   59   3 15 15   -13.92
 1  3   59   3 15 25   -13.42
 1  4   59   3 15 35   -12.90
 1  5   59   3 15 45   -14.11
 1  6   59   3 15 55   -13.17
 1  7   59   3 16  5   -11.12
 1  8   59   3 16 15    -8.65
 1  9   59   3 16 25    -6.98
 1 10   59   3 16 35    -5.90
 1 11   59   3 16 45    -6.03
 1 12   59   3 16 55    -8.26
 1 13   59   3 17  5   -11.16
 1 14   59   3 17 15   -12.73
 1 15   59   3 17 25   -11.05


30 29   61 57 19 45    -7.33
30 30   61 57 19 55   -13.33
30 31   61 57 20  5   -16.33
30 32   61 57 20 15   -21.33
30 33   61 57 20 25   -27.33
30 34   61 57 20 35   -33.33
30 35   61 57 20 45   -35.33
30 36   61 57 20 55   -36.33
```

```
6' BY 10' MEAN GRAVITY ANOMALIES IN
CENTRAL SWEDEN

Area: min/max latitude:       59/62 degree
      min/max longitude:      15/21 degree
Total number of data:         1080
Reference field:              GRS 1980
Unit of gravity anomalies:    mGal
File name:                    GRAV.DAT

All coordinates refer to the block centers
Format(2I3,1X,4I3,F8.2)

(H. Fan, Stockholm, 1990-02-19, 7:17pm)
```

Non-classic theories

- Gravity reductions – classic theory does not fit the reality
- Molodenskii's theory
- Bjerhammar's methods
- Least squares collocation
- *Global gravity field in spherical harmonic expansions*
- *Truncation of Stokes' formula, combination of terrestrial gravity data and global data sets*
- *Dedicated satellite gravity misisons for determination of GGMs*

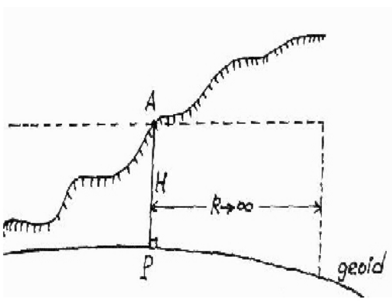
Why gravity reduction ?

- Assumption in Stokes's theory: the disturbing potential T is harmonic on/outside the geoid. This requires that there are no masses outside the geoid
- Objectives of **gravity reduction**:
  - ✓ remove masses outside the geoid
  - ✓ compute gravity on the geoid from gravity measurements made on the earth's surface
- Gravity reduction causes the extermal gravity field of the earth to change, *indirect effects*
- In geophysics/geology, one uses gravity reduction to remove systematic effect of the topography

Free-air gravity reduction

Free-air = there is no mass between P and A

$$g_A = g_P + \frac{\partial g}{\partial H}H + \frac{1}{2!}\frac{\partial^2 g}{\partial H^2}H^2 + \frac{1}{3!}\frac{\partial^3 g}{\partial H^3}H^3 + \cdots \approx g_P + \frac{\partial g}{\partial H}H$$



$$g_P \approx g_A + \delta_F$$

Free-air correction

$$\delta_F = -\frac{\partial g}{\partial H}H \approx +0.3086\ H^{\,metre} \quad (mGal)$$

$$\Delta g_F = g_P - \gamma_Q = g_A - \gamma_Q + \delta_F$$

Free-air gravity anomaly

Bouguer gravity reduction

Topography is like a Bouguer plate (big cylinder)

$$B = 2\pi G\rho_0\, H = +0.1119\, H^{\,metre} \quad (mGal)$$

$$g_P = g_A + \delta_F - B$$

$$\delta_B = \delta_F - B = 0.3086\, H - 0.1119\, H = +0.1967\, H^{\,metre} \quad (mGal)$$

Bouguer gravity correction

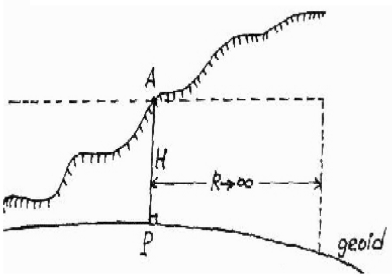


Figure 3.12: Bouguer gravity reduction

$$\Delta g_B = g_P - \gamma_Q = g_A - \gamma_Q + \delta_B$$

Bouguer gravity anomaly

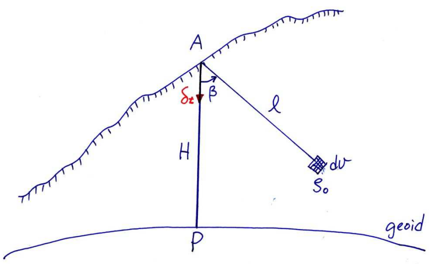
Topographic gravity reduction

(total) topographic correction:

Vertical gravitational force at A by the total topography:

$$\delta_t = G \iiint_{v_t} \frac{\cos\beta\, \rho_0\, dv}{\ell^2}$$

Standard density is used:
ρ_0 = 2.67 g/cm**3



$$g_P = g_A + \delta_F - \delta_t$$

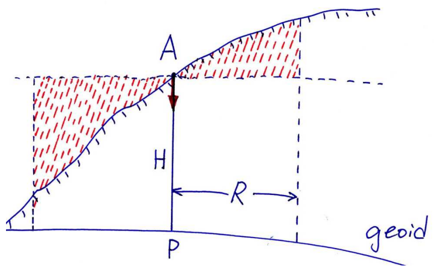
$$\delta_T = \delta_F - \delta_t$$

topographic reduction

$$\Delta g_T = g_P - \gamma_Q = g_A - \gamma_Q + \delta_T$$

topographic gravity anomaly

Topographic correction in 2 steps



- Bouguer correction
- Terrain correction:
  vertical gravitational force at A by the extra masses above A and mass deficit (air) below A

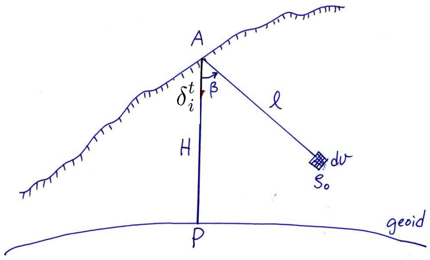
  Always postive !
- Total topographic correction =

  = Bouguer correction + terrain

Topographic-isostatic reduction

= topographic gravity reduction using varying crustal density calculated from isostatic theory with isostatic compensation

$$\delta_i^t = G \iiint_{v_t} \frac{\cos\beta \, \rho_{\_} \, dv}{\ell^2}$$



Topographic-isostatic reduction

$$\delta_I = \delta_F - \delta_i$$

$$\delta_i = \delta_i^t - \delta_i^c$$

Isostatic compensation

Topographic-isostatic gravity anamol

$$\Delta g_T = g_P - \gamma_Q = g_A - \gamma_Q + \delta_I$$

Isostatic theory

- The earth's surface layer consists of extra masses in the mountains and mass deficit in the oceans
- The isostatic theory says that the masses *somehow* are kept in balance everywhere
- Isostatic phenomena has been evidenced and accepted, though there exist differing **mathematical models** to describe this phenomena
- Development of isostacy helped to understand the visco-elastic character of the earth's upper mantle

Pratt-Hayford model

The model of "*Baking bread*"

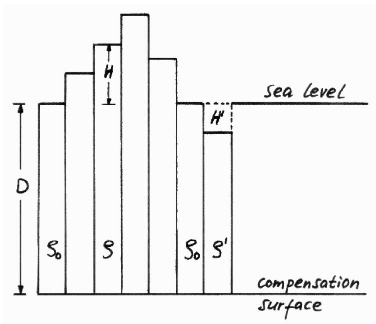


Figure 3.13: Isostatic model of Pratt-Hayford

D=100km = compensation depth

On land

$$D \cdot \rho_0 = (D + H) \cdot \rho \quad \text{or} \quad \rho = \frac{D}{D+H} \cdot \rho_0$$

On oceans

$$D \cdot \rho_0 = (D - H') \cdot \rho' + H' \cdot \rho_w$$

$$\text{or} \quad \rho' = \frac{D \cdot \rho_0 - H' \cdot \rho_w}{D - H'}$$

Total topographic-isostatic correction

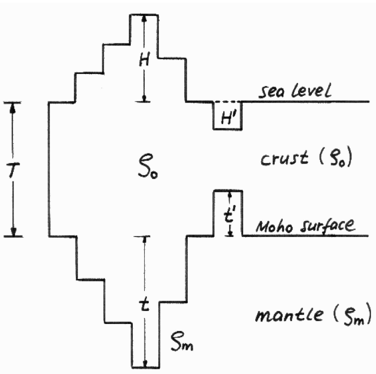
$$\delta_i = \delta_i^t - \delta_i^c$$

Isostatic compensation

Airy-Heiskanen model

Isberg floating in the water

∾ Crust floating in the upper mantle



T=30km = crustal depth

On land

$$H \cdot \rho_0 = t \cdot (\rho_m - \rho_0)$$

$$\text{or} \quad t = \frac{\rho_0}{\rho_m - \rho_0} \cdot H \approx 4.45\ H$$

On oceans

$$H' \cdot (\rho_0 - \rho_w) = t' \cdot (\rho_m - \rho_0)$$

$$\text{or} \quad t' = \frac{\rho_0 - \rho_w}{\rho_m - \rho_0} \cdot H' \approx 2.78\ H'$$

Other reduction methods

- **Helmert condensation method**: Topographic masses are condensed into a surface layer *on the geoid* with density μ :

$$\mu = \rho \cdot H$$

Total mass of the earth is unchanged. But external gravity field is changed, causing the *indirect effect*

- **Rudzki reduction**: the topographic mass element *dm* is relocated as *dm' below the geoid*:

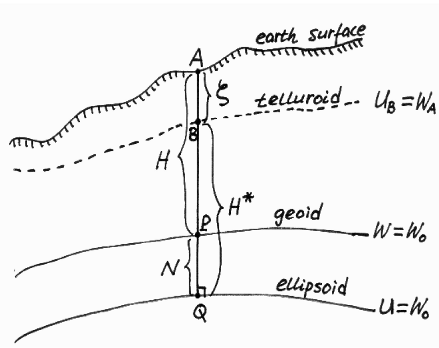
$$r' = \frac{R^2}{r} \quad \text{and} \quad dm' = \frac{R}{r} dm$$

Geopotential on the geoid is unchanged. But external gravity field is changed.

Direct and indicrect effects

- The topographic (and isostatic) effect on the gravity anomaly is called the *direct efffect*
- Change in the geoid height due to the direct effect of the topography is called the *indirect effect*
- The atmosphere of the earth behaves in a similar way as the topography, thus causing both *direct* and *indirect* effects in geoidal determination

Molodenskii's theory



Telluroid is defined by:

$$W_A = U_B$$

Separation: telluroid-surface:

$$\zeta$$

Disturbing potential:

$$T_A = W_A - U_A$$

Gravity anomaly:

$$\Delta g_A = g_A - \gamma_B$$

$$C_A = W_0 - W_A = U_Q - U_B = \int_{Q \to P \to B} \gamma \cdot dh = \overline{\gamma}_A \cdot H^*$$

Molodenskii's gravity anomaly

$$\Delta g_A = g_A - \gamma_B$$

$$\gamma_B = \gamma_Q - 2\gamma_e \cdot \left(\frac{H_A}{a}\right) \cdot \left[1 + f + m + \left(-3f + \frac{5}{2}m\right) \cdot \sin^2\phi\right] + 3\gamma_e \left(\frac{H_A}{a}\right)^2$$

a = equatorial radius

γ_e = normal gravity on the equator

f = geometric flattening of the reference ellipsoid

$m = \frac{w^2a^2b}{GM} = 0.00344978600308$ (for GRS 80)

ϕ = geodetic latitude of P

$$\Delta g_A = g_A - \gamma_Q + 2\gamma_e \cdot \left(\frac{H_A}{a}\right) \cdot \left[1 + f + m + \left(-3f + \frac{5}{2}m\right) \cdot \sin^2\phi\right] - 3\gamma_e \left(\frac{H_A}{a}\right)^2$$

Basic equations

$$T_A = W_A - U_A = U_B - U_A =$$

$$= U_B - \left(U_B + \frac{\partial U}{\partial n'}\zeta + \frac{1}{2!}\frac{\partial U}{\partial n'}\zeta^2 + \cdots\right) \approx -\frac{\partial U}{\partial n'}\zeta = \gamma_B \zeta$$

$$\zeta = \frac{T_A}{\gamma_B}$$

$$\Delta g_A = g_A - \gamma_B = \left(-\frac{\partial W_A}{\partial h}\right) - \left(-\frac{\partial U_B}{\partial n'}\right)$$

$$= -\frac{\partial T_A}{\partial h} - \frac{\partial U_A}{\partial h} + \frac{\partial U_B}{\partial n'} \approx -\frac{\partial T_A}{\partial h} + \gamma_A - \gamma_B$$

$$\approx -\frac{\partial T_A}{\partial h} + \left(\gamma_B + \frac{\partial \gamma}{\partial n'}\zeta + \cdots\right) - \gamma_B \approx -\frac{\partial T_A}{\partial h} + \frac{1}{\gamma_B}\frac{\partial \gamma}{\partial n'} \cdot T_A$$

$$\Delta g_A = -\frac{\partial T_A}{\partial r} - \frac{2}{r}T_A$$

Molodenskii's solution

$$\Delta\left(T_A\right)=0$$

$$\Delta g_A=-\frac{\partial T_A}{\partial r}-\frac{2}{r}T_A$$

$$T_A=G\iint_S\frac{\mu\cdot dS}{\ell}$$

$$T_A=\sum_{i=0}^{\infty}T_A^{(i)}=T_A^{(0)}+T_A^{(1)}+T_A^{(2)}+\cdots$$

$$\zeta=\frac{T_A}{\gamma_B}=\sum_{i=0}^{\infty}\zeta^{(i)}=\zeta^{(0)}+\zeta^{(1)}+\zeta^{(2)}+\cdots$$

$$\zeta^{(0)}=\frac{R}{4\pi\gamma_B}\iint_\sigma S\left(\psi\right)\cdot G_0\cdot d\sigma\ ,\quad G_0=\Delta g_A$$

$$\zeta^{(1)}=\frac{R}{4\pi\gamma_B}\iint_\sigma S\left(\psi\right)\cdot G_1\cdot d\sigma$$

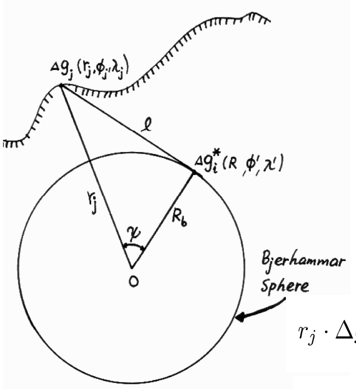
$$G_1=-h\frac{R^2}{2\pi}\iint_\sigma\frac{h-h_p}{\ell_0}\left(\Delta g_A+\frac{3\gamma_B}{2R}\zeta^{(0)}\right)\cdot d\sigma$$

Discrete BVP of Bjerhammar

- Given a finite set of discrete gravity measurements,
- we look for an estimate of the disturbing potential
- which satisfies the boundary value condition at the discrete measurement points:

$$-\frac{\partial\widehat{T}_j}{\partial r_j}-\frac{2}{r_j}\widehat{T}_j=\Delta g_j\ ,\qquad j=1,2,3,\cdots,\ n$$

Bjerhammar's method



$\Delta g(r_j, \phi, \lambda)$	not harmonic
$r_j \cdot \Delta g(r_j, \phi, \lambda)$	harmonic under spherical approximation

Apply Poisson's integral on $r_j \cdot \Delta g(r_j, \phi, \lambda)$

$$r_j \cdot \Delta g(r_j, \phi, \lambda) = \frac{R_b(r_j^2 - R_b^2)}{4\pi} \iint_\sigma \frac{(r \cdot \Delta g^*)_{r=R_b}}{\ell^3} \, d\sigma(\phi', \lambda')$$

$$\ell = \sqrt{r_j^2 + R_b^2 - 2 r_j R_b \cos\psi}$$

$$\Delta g(r_j, \phi, \lambda) = \frac{R_b^2(r_j^2 - R_b^2)}{4\pi r_j} \iint_\sigma \frac{\Delta g^*(R_b, \phi', \lambda')}{\ell^3} \, d\sigma(\phi', \lambda')$$

Bjerhammar's solution

$$\Delta g(r_j, \phi, \lambda) = \sum_{i=1}^{m} \left(\frac{R_b^2(r_j^2 - R_b^2)}{4\pi r_j} \iint_{\sigma_i} \frac{\Delta g_i^*(R_b, \phi', \lambda')}{\ell_{ji}^3} \, d\sigma(\phi', \lambda') \right) = \sum_{i=1}^{m} (a_{ji} \cdot \Delta g_i^*)$$

$$a_{ji} = \frac{R_b^2(r_j^2 - R_b^2)}{4\pi r_j} \frac{A_i}{\ell_{ji}^3}$$

$$\underset{n \cdot 1}{\Delta g} = \underset{n \cdot m}{A} \cdot \underset{m \cdot 1}{\Delta g^*}$$

$$\widehat{T}(r_j, \phi, \lambda) = \frac{R}{4\pi} \iint_\sigma \Delta g_i^* \, S\left(r_i, \psi_{ji}\right) \, d\sigma_i$$

$$N(r_j, \phi, \lambda) = \frac{\widehat{T}}{\gamma} = \frac{R}{4\pi\gamma} \iint_\sigma \Delta g_i^* \, S\left(r_i, \psi_{ji}\right) \, d\sigma_i$$

$$\begin{bmatrix} \xi_j \\ \eta_j \end{bmatrix} = \frac{1}{4\pi\gamma} \iint_\sigma \Delta g_i^* \frac{dS(r_i, \psi_{ji})}{d\psi_{ji}} \begin{bmatrix} \cos\alpha_{ji} \\ \sin\alpha_{ji} \end{bmatrix} d\sigma_i$$

Least squares collocation

Given a set of discrete measurements:

gravity anomaly Δg_i at n points $(i = 1,\ 2,\ 3 \cdots n)$

We look for a ***linear*** estimate of the disturbing potential

$$\widehat{T}_p = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n) \cdot \begin{bmatrix} \Delta g_1 \\ \Delta g_2 \\ \cdots \\ \Delta g_n \end{bmatrix} = \alpha^T \cdot \Delta g$$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix}, \quad \Delta g = \begin{bmatrix} \Delta g_1 \\ \Delta g_2 \\ \cdots \\ \Delta g_n \end{bmatrix}$$

Unknown cofficients to be determined

Error of the prediction

$$\varepsilon_p = \widehat{T}_p - T_p$$

$$m_\varepsilon^2 = M\left(\varepsilon_p^2\right)$$

A homogeneous, isotropic, averaging operator

$$M(\cdot) = \frac{1}{8\pi^2} \int_{\lambda_p=0}^{2\pi} \int_{\phi_p=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\alpha_{pq}=0}^{2\pi} (\cdot)\ d\alpha_{pq}\ d\sigma_{pq}$$

Global mean square error of the prediction

$$m_\varepsilon^2 = M\left(\varepsilon_p^2\right) = M\left(\left[\alpha^T \Delta g - T_p\right]\left[\alpha^T \Delta g - T_p\right]^T\right) = \alpha^T C_{gg} \alpha - 2 C_{Tg} \alpha + m_T^2$$

Variance-covariance matrices

Variance matrix of the gravity anomaly:

$$C_{gg} = M\left(\Delta g \Delta g^T\right) = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}, \quad c_{ij} = M\left(\Delta g_i \Delta g_j\right)$$

Covariance matrix between disturbing potential and the gravity anomaly:

$$C_{Tg} = M\left(T_p \Delta g^T\right) = \left(c_{p1} \;\; c_{p2} \;\; \cdots \;\; c_{pn}\right), \quad c_{pj} = M\left(T_p \Delta g_j\right)$$

Variance of the disturbing potential:

$$m_T^2 = M\left(T_p^2\right)$$

Least squares estimate

$$\frac{d(m_\varepsilon^2)}{d\alpha} = 2\alpha^T C_{gg} - 2C_{Tg} = 0$$

$$\longrightarrow \quad \alpha^T = C_{Tg} \cdot C_{gg}^{-1}$$

$$\widehat{T}_p = C_{Tg} \cdot C_{gg}^{-1} \cdot \Delta g$$

$$m_\varepsilon^2 = m_T^2 - C_{Tg} \cdot C_{g\,g}^{-1} \cdot C_{g\,T}$$

Problems and forthcoming lectures

- Global gravity field - Global gravitational models (GGM)
- Truncation error in Stokes' formula
- Combination of ground measurements and GGMs
- Effects of topography, atmosphere and ellipticity
- Dedicated satellite gravity missions