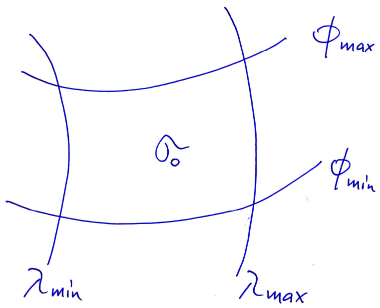
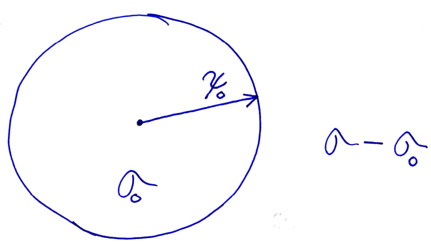
Geoid determination by KTH-method

- Truncation errors in Stokes's formula
- A simple combination method
- Least Squares Modification of Stokes' formula with Additive corrections (LSMSA) (the **KTH method**)
- Computation of the **NKG2015** geoid model

Data area with gravity measurements





σ_0 is an equi-angular block

σ_0 is a spherical cap of radius ψ_0 (spherical distance/angular distance)

Truncation error of Stokes' formula

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma} S(\psi) \cdot \Delta g(R, \phi', \lambda') \cdot d\sigma(\phi', \lambda')$$

$$\widehat{N} = \frac{R}{4\pi\gamma} \iint_{\sigma_0} S(\psi) \cdot \Delta g \cdot d\sigma$$

← Stokes' formula is truncated

$$\delta N = \frac{R}{4\pi\gamma} \iint_{\sigma_0} S(\psi) \cdot \Delta g \cdot d\sigma - \frac{R}{4\pi\gamma} \iint_{\sigma} S(\psi) \cdot \Delta g \cdot d\sigma$$

$$= -\frac{R}{4\pi\gamma} \iint_{\sigma - \sigma_0} S(\psi) \cdot \Delta g \cdot d\sigma$$

→ Truncation error of Stokes' formula at one point

Truncated Stokes' function

$$\bar{S}(\psi) = \begin{cases} 0 & \text{for } \psi < \psi_0 \\ S(\psi) & \text{for } \psi \geq \psi_0 \end{cases}$$

$$\bar{S}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n P_n(cos\psi), \quad \text{for } \ 0 \leq \psi \leq \pi$$

$$Q_n(\psi_0) = \int_{\psi_0}^{\pi} S(\psi) P_n(cos\psi) \sin\psi d\psi$$

↘ Molodenski truncation coefficient

Truncation error

$$\delta N = -\frac{R}{4\pi\gamma}\iint_{\sigma-\sigma_0} S(\psi)\cdot\Delta g\cdot d\sigma$$

$$= -\frac{R}{4\pi\gamma}\iint_{\sigma} \overline{S}(\psi)\cdot\Delta g\cdot d\sigma = -\frac{R}{2\gamma}\sum_{n=2}^{\infty} Q_n \Delta g_n$$

$$\|\delta N\|^2 = \frac{1}{4\pi}\iint_{\sigma} \delta N^2 d\sigma = c^2 \sum_{n=2}^{\infty} Q_n^2 \cdot c_n$$

global mean square value of truncation error of Stokes' formula

Computation of Molodenskii truncation coefficients

$$Q_n(\psi_0) = -\frac{1}{(n-1)(n+2)}\Big\{n\, S(t)\left[P_{n-1}(t) - tP_n(t)\right] - (1-t^2)P_n(t)\frac{dS(t)}{dt} + $$

$$+ 2I_n(t) + 9J_n(t) + 2K_n(t)\Big\}$$

$$t = \cos\psi_0$$

$$S(t) = \sqrt{\frac{2}{1-t}} - 6\sqrt{\frac{1-t}{2}} + 1 - 5t - 3t\ln$$

$$\frac{dS(t)}{dt} = -8 + \frac{3\sqrt{2}}{\sqrt{1-t}} + \frac{1}{\sqrt{2}(1-t)^{3/2}} + \frac{3\left(\sqrt{2}-\sqrt{1-t}\right)}{\sqrt{2}(1-t^2)}$$

$$-\,3\,\ln\left(\sqrt{\frac{1-t}{2}} + \frac{1-t}{2}\right)$$

Computation of Molodenskii truncation coefficients

$$I_n(t) = \int_{\psi_0}^{\pi} P_n(\cos\psi)\ \sin\psi\ d\psi = \frac{1}{2n+1}\{\ P_{n+1}(t) - P_{n-1}(t)\ \}$$

$$J_n(t) = \int_{\psi_0}^{\pi} \cos\psi P_n(\cos\psi)\sin\psi d\psi$$

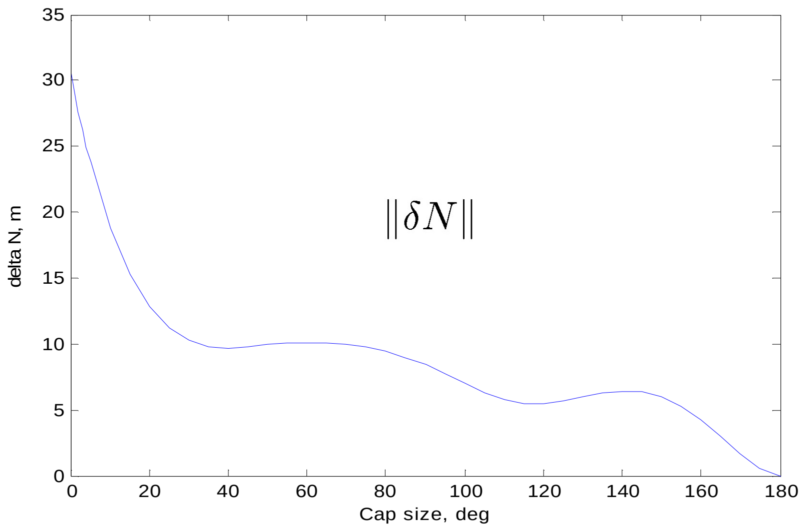
$$= \frac{1}{2n+1}\left\{\frac{n+1}{2n+3}P_{n+2}(t) + \frac{2n+1}{(2n-1)(2n+3)}P_n(t) - \frac{n}{2n-1}P_{n-2}(t)\right\}$$

$$K_n(t) = (8)^{-\frac{1}{2}}\int_{\psi_0}^{\pi}(1-\cos\psi)^{-\frac{3}{2}}\,P_n(\cos\psi)\sin\psi d\psi$$

$$= 2K_{n-1}(t) - K_{n-2}(t) - \frac{P_n(t) - P_{n-2}(t)}{(2n-1)\sqrt{2(1-t)}}$$

$$K_0(t) = -\frac{1}{2} + \sqrt{\frac{1}{2(1-t)}},\quad K_1(t) = -\frac{3}{2} + \sqrt{\frac{1}{2(1-t)}} + \sqrt{\frac{1-t}{2}}$$

Truncation error of Stokes’ formula



Degree variance of gravity anomaly

$$\Delta g(R,\overline{\phi},\lambda) = \frac{GM}{R^2}\sum_{n=2}^{\infty}\left\{(n-1)\sum_{m=-n}^{n}\overline{C}_{nm}\overline{Y}_{nm}(\overline{\phi},\lambda)\right\} = \sum_{n=2}^{\infty}\Delta g_n(R,\overline{\phi},\lambda)$$

$$\Delta g_n(R,\overline{\phi},\lambda) = \frac{GM}{R^2}(n-1)\sum_{m=-n}^{n}\overline{C}_{nm}\overline{Y}_{nm}(\overline{\phi},\lambda)$$

global mean square value of n-th degree gravity anomay:

$$c_n = \frac{1}{4\pi}\iint_{\sigma}(\Delta g_n)^2\, d\sigma = \left(\frac{GM}{R^2}\right)^2 (n-1)^2\sum_{m=-n}^{n}(\overline{C}_{nm})^2$$

→ Gravity anomaly degree variance for degree n

Tscherning-Rapp model (1974):

$$c_n = 425.28\ (mGal)^2\ \frac{n-1}{(n-2)\ (n+24)}\ t^{n+2}, \quad t = 0.999617$$

Geoid height error from GGM

$$N(R,\overline{\phi},\ \lambda) = \frac{GM}{R\,\gamma}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\overline{C}_{nm}\ \overline{Y}_{nm}(\overline{\phi},\ \lambda)$$

$$\widehat{N}(R,\overline{\phi},\ \lambda) = \frac{GM}{R\,\gamma}\sum_{n=2}^{L}\sum_{m=-n}^{n}\overline{C}'_{nm}\ \overline{Y}_{nm}(\overline{\phi},\ \lambda)$$

$$\delta\overline{C}_{nm} = \overline{C}'_{nm} - \overline{C}_{nm}$$

$$\delta N = \widehat{N} - N = \frac{GM}{R\,\gamma}\sum_{n=2}^{L}\sum_{m=-n}^{n}\delta\overline{C}_{nm}\ \overline{Y}_{nm}(\overline{\phi},\ \lambda) - \frac{GM}{R\,\gamma}\sum_{n=L+1}^{\infty}\sum_{m=-n}^{n}\overline{C}_{nm}\ \overline{Y}_{nm}(\overline{\phi},\ \lambda)$$

$$\|\delta N\|^2 = \frac{1}{4\pi}\iint_{\sigma}E\left(\delta N^2\right)\, d\sigma = \left(\frac{R}{2\gamma}\right)^2\left\{\sum_{n=2}^{L}\left[\left(\frac{2}{n-1}\right)^2 dc_n\right] + \sum_{n=L+1}^{\infty}\left[\left(\frac{2}{n-1}\right)^2 c_n\right]\right\}$$

$$dc_n = \left(\frac{GM}{R^2}\right)^2 (n-1)^2\sum_{m=-n}^{n}E\left[\left(\delta\overline{C}_{nm}\right)^2\right]$$

Gravity anomaly **error degree variance** of degree n for GGM

A simple combination method

$$N(R,\phi,\lambda) = \frac{R}{4\pi\gamma}\iint_\sigma \Delta g\, S(\psi)\, d\sigma$$

$$= \frac{R}{4\pi\gamma}\iint_\sigma S(\psi)\left(\Delta g - \Delta\widehat{g}_M\right)\, d\sigma + \frac{R}{4\pi\gamma}\iint_\sigma S(\psi)\Delta\widehat{g}_M\, d\sigma$$

$$= \frac{R}{4\pi\gamma}\iint_\sigma S(\psi)\left(\Delta g - \Delta\widehat{g}_M\right)\, d\sigma + \widehat{N}_M(\phi,\lambda)$$

The above result inspired us to propose an estimate:

$$\widehat{N}(\phi,\lambda) = \frac{R}{4\pi\gamma}\iint_{\sigma_0} S(\psi)\left(\Delta g' - \Delta\widehat{g}_M\right)\, d\sigma + \widehat{N}_M(\phi,\lambda)$$

$$\widehat{N}(\phi,\lambda) = \frac{R}{4\pi\gamma}\iint_{\sigma_0} S(\psi)\,\Delta g'\, d\sigma + \frac{R}{4\pi\gamma}\iint_{\sigma-\sigma_0} S(\psi)\,\Delta\widehat{g}_M\, d\sigma$$

Error estimate of the new method

$$\|\,\delta N\,\|^2 = \frac{1}{4\pi}\iint_\sigma E\left(\widehat{N} - N\right)^2 d\sigma$$

$$= c^2\left\{\sum_{n=2}^{n_{\max}} Q_n^2 \cdot dc_n + \sum_{n=M+1}^{\infty} Q_n^2 \cdot c_n + \sum_{n=2}^{\infty}\left(\frac{2}{n-1} - Q_n\right)^2 \sigma_n^2\right\}$$

$c = R/(2\gamma)$

$$dc_n = \left(\frac{GM}{R}\right)^2\left(\frac{n-1}{R}\right)^2\sum_{m=-n}^{n} E\left\{\left(\widehat{C}_{nm} - \overline{C}_{nm}\right)^2\right\}$$

$$c_n = \left(\frac{GM}{R}\right)^2\left(\frac{n-1}{R}\right)^2\sum_{m=-n}^{n} \overline{C}_{nm}^2$$

$$\sigma_n^2 = E\left\{(\Delta g_n' - \Delta g_n)^2\right\}$$

$\Delta g_n'$ and Δg_n are the n-th degree Laplace harmonics of $\Delta g'$ and Δg

Least squares modification method

$$\tilde{N} = \frac{c}{2\pi}\iint_{\sigma_o} S^L(\psi)\Delta g d\sigma + c\sum_{n=2}^{M} b_n \Delta g_n^{EGM}$$

$$S^L(\psi) = \sum_{n=2}^{\infty}\frac{2n+1}{n-1}P_n(\cos\psi) - \sum_{n=2}^{L}\frac{2n+1}{2}s_n P_n(\cos\psi)$$

$$b_n = \left(Q_n^L + s_n^*\right)\frac{c_n}{c_n + dc_n} \quad for \;\; 2 \le n \le M \,, \quad c = R/2\gamma \qquad s_n^* = \begin{cases} s_n & if \;\; 2 \le n \le L \\ 0 & otherwise \end{cases}$$

$$Q_n^L = Q_n - \sum_{k=2}^{L}\frac{2k+1}{2}s_k e_{nk}$$

$$e_{nk}\left(\psi_0\right) = \int_{\psi^0}^{\pi} P_n(\cos\psi)P_k(\cos\psi)\sin\psi d\psi$$

Least squares solution of s_n

$$\sum_{r=2}^{L} a_{kr}s_r = h_k, \quad k = 2,3,\ldots..,L,$$

$$a_{kr} = \sum_{n=2}^{\infty} E_{nk}E_{nr}C_n + \delta_{kr}C_r - E_{kr}C_k - E_{kr}C_r,$$

$$h_k = \Omega_k - Q_k C_k + \sum_{n=2}^{\infty}\left(Q_n C_n - \Omega_k\right)E_{nk}$$

$$\Omega_k = \frac{2\sigma_k^2}{k-1}$$

$$\delta_{kr} = \begin{cases} 1 & if\; k = r \\ 0 & otherwise \end{cases}$$

$$C_k = \sigma_k^2 + \begin{cases} c_k dc_k / \left(c_k + dc_k\right) & if \;\; 2 \le k \le M \\ c_k & if \;\; k > M \end{cases}$$

$$E_{nk} = \frac{2k+1}{2}e_{nk}\left(\psi_0\right)$$

Geoid height with additive corrections

Geoid height from LSMM

$$\hat{N} = \tilde{N} + \delta N_{comb}^{Topo} + \delta N_{DWC} + \delta N_{comb}^{a} + \delta N_{e}$$

Topographic correction

Atmospheric correction

Ellipsoidal correction

Down**W**ard **C**ontinuation correction

Combined topographic correction

$$\delta N_{comb}^{Topo} = \delta N_{dir} + \delta N_{indir} \approx -\frac{2\pi G\rho}{\gamma} H^2$$

Downward continuation correction

$$\delta N_{DWC} = \frac{c}{2\pi}\iint_{\sigma_0} S_L(\psi)\left(\Delta g^* - \Delta g\right) d\sigma,$$
$$= \delta N_{DWC}^{(1)}(P) + \delta N_{DWC}^{L1,Far}(P) + \delta N_{DWC}^{L2}(P),$$

$$\delta N_{DWC}^{(1)}(P) = H_P\left(\frac{\Delta g(P)}{\gamma} + 3\frac{N_P^0}{r_P} - \frac{1}{2\gamma}\left.\frac{\partial \Delta g}{\partial r}\right|_P H_P\right),$$

$$\delta N_{DWC}^{L1,Far}(P) = c\sum_{n=2}^{M}\left(s_n^* + Q_n^L\right)\left[\left(\frac{R}{r_P}\right)^{n+2} - 1\right]\Delta g_n(P)$$

$$\delta N_{DWC}^{L2}(P) = \frac{c}{2\pi}\iint_{\sigma_0} S_L(\psi)\left(\left.\frac{\partial \Delta g}{\partial r}\right|_P\right)\left(H_P - H_Q\right) d\sigma_Q$$

$$\left.\frac{\partial \Delta g}{\partial r}\right|_P = \frac{R^2}{2\pi}\iint_{\sigma_0}\frac{\Delta g_Q - \Delta g_P}{l_0^3} d\sigma_Q - \frac{2}{R}\Delta g(P), \qquad l_o = 2R\sin\frac{\psi_{PQ}}{2}$$

Atmospheric/ellipsoidal corrections

$$\delta N_{comb}^a(P) = -\frac{2\pi R\rho_0}{\gamma}\sum_{n=2}^{M}\left(\frac{2}{n-1} - s_n - Q_n^L\right)H_n(P)$$
$$-\frac{2\pi R\rho_0}{\gamma}\sum_{n=M+1}^{\infty}\left(\frac{2}{n-1} - \frac{n+2}{2n+1}Q_n^L\right)H_n(P),$$

where ρ_0 is the density at sea level radius ρ^0 $\left(\rho^0 = 1.23\times10^{-3}\, g\, cm^{-3}\right)$ multiplied by the gravitational constant G $\left(G = 6.673\times10^{-11}\, m^3 kg^{-1} s^{-2}\right)$,

$$\delta N_e \approx \psi_0^o\left[\left(0.12 - 0.38\cos^2\theta\right)\Delta g + 0.17\tilde{N}\sin^2\theta\right]_{mm}$$

Practical geoid computation

- Collect gravity data: validation, griding
- Select GGM. Verification: GPS-levelling, g-data
- Collect DTMs. SRTM: 3′′ x 3′′, 1′ x 1′ , 5′ x 5′
- Choose method. KTH: LSMM with additive corrections
- Verification: GPS-levelling, astro-levelling
- Fitting to WGS 84 (3-7par): corrective geoid model
- Updating: new data/method

Geoid research at KTH Geodesy

- Started in 1960's with Prof. Arne Bjerhammar
- His discrete methods and *Bjerhammar sphere* are widely recognised and won him the Gauss Prize
- In 1984, Lars Sjöberg proposed the least squares modification of Stokes' formula to combine terrestrial gravity data with GGMs
- Since 1985, 10 PhD projects are started to the development of the *KTH method* for precise geoid determination

Nordic Geodetic Commission (NKG)

- The Nordic Geodetic Commission (NKG) is an association of geodesists from Denmark, Finland, Iceland, Norway and Sweden.
- The purpose of NKG is mutual exchange, cooperation and coordination of work with the geodetic infrastructure in the region.
- The Baltic countries Estonia, Latvia and Lithuania are also included in NKG
- NKG has computed several geoid models:
  - NKG1986 (Tscherning and Forsberg 1986)
  - NKG1989 (Forsberg 1990)
  - NKG1996 (Forsberg et al. 1996)
  - NKG2002
  - NKG2004
  - NKG2015
- These NKG models have been used to construct national height corrections models.

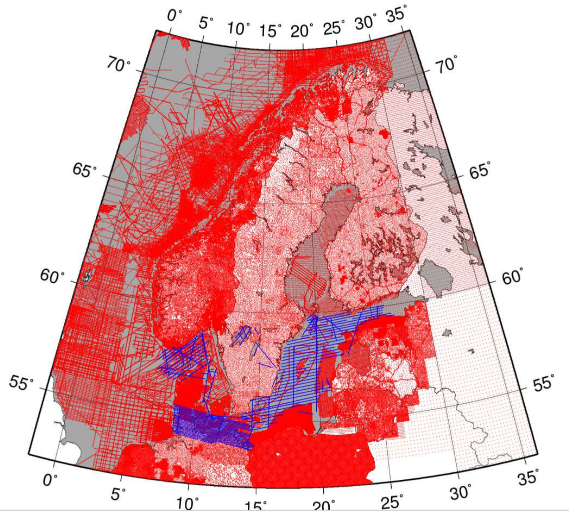
NKG2015 project

- Started in 2011 as a Nordic/Baltic cooperation project
- Common technical specifications: reference systems/frames, permanent tide, postglacial land uplift epoch, etc.
- Update databases: gravity, DEM, GNSS/levelling
- Main features:
  - ✓ in zero-tidal system
  - ✓ reference epoch 2000.0
  - ✓ GGM: GO_CONS_GCF_2_DIR_R5

NKG2015 accuracy: 1.5-2 cm on land; 1 cm in smooth areas with good data

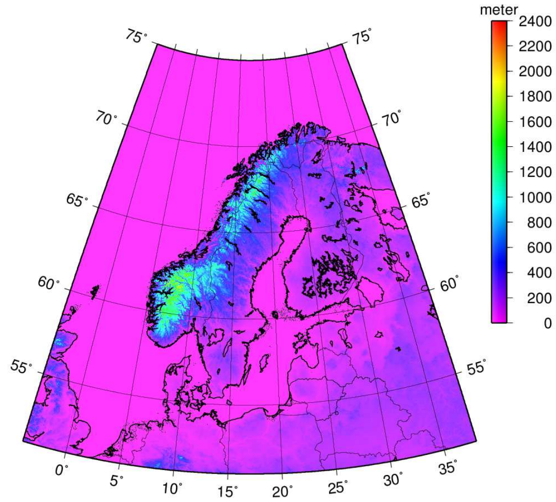
- KTH's LSMSA and Danish RCR-method has been tested against GNSS/levelling. The KTH method was chosen to be the method for the final NKG2015 geoid model (Source: **Ågren 2016**)

Gravity data



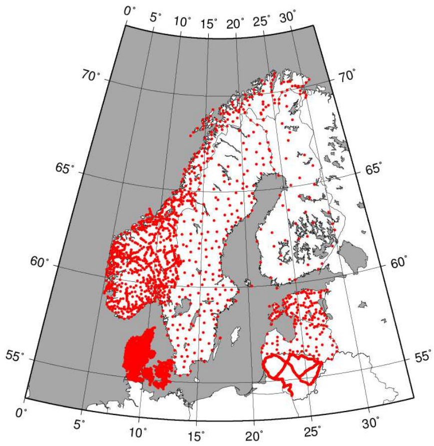
- Horizontal positions and heights transformed to national ETRS 89 and EVRS realisations.
- Updated with quality checked and new data from all the Nordic and Baltic countries.
- Data transformed to zero permanent tide system and epoch 2000.0 (if meaningful considering the accuracy, etc.).

Digital Elevation Models (DEM)



- 3″x3″ resolution
- The Nordic part of the EGG08 DTM complemented with new national models from Finland, Estonia, Latvia and Lithuania.
- National DEMs in all the Nordic/Baltic countries and SRTM3 or shifted GTOPO30 elsewhere.

GNSS/levelling data



- Levelled heights in national EVRS realisations.
- The GNSS-heights have been transformed to the common ETRS 89 realisation ETRF2000 epoch 2000.0 using the so-called NKG2008 transformations (Häkli et al. 2016).
- These transformations consist of a 7-parameter part and a postglacial land uplift correction part (NKG2005LU).

Testing against GNSS-levelling

Centre	Sol	Method	Standard deviation in 1-parameter fit (meter)								Comment
			All	Denmark	Estonia	Finland	Latvia	Lithuania	Norway	Sweden	
NKG2015		LSMSA, unbiased, KTH/LM	0.0285	0.0168	0.0147	0.0215	0.0246	0.0333	0.0285	0.0186	FINAL MODEL, ver. 4 gravity with updates as decided in WG meeting. Improved gridding in Norway
LM	1a	LSMSA, unbiased, KTH/LM	0.0299	0.0169	0.0147	0.0214	0.0247	0.0332	0.0333	0.0185	
TUT/ELB	2a	LSMSA, biased, TUT	0.0362	0.0165	0.0192	0.0233	0.0253	0.0347	0.0421	0.0238	No ice corr, small area
	2b	LSMSA, unbiased, TUT	0.0381	0.0198	0.0113	0.0211	0.0238	0.0311	0.0458	0.0244	No ice corr, small area
	2c	LSMSA, unbiased, TUT	0.0383	0.0200	0.0112	0.0211	0.0240	0.0312	0.0463	0.0244	No ice corr, small area
	2d	LSMSA, unbiased, TUT	0.0373	0.0192	0.0120	0.0203	0.0245	0.0316	0.0443	0.0241	No ice corr, small area
	2e	LSMSA, unbiased, KTH/LM	0.0319	0.0205	0.0114	0.0258	0.0228	0.0314	0.0387	0.0210	No ice corr
	2f	LSMSA, unbiased, KTH/LM	0.0309	0.0200	0.0120	0.0257	0.0236	0.0317	0.0365	0.0211	No ice corr
	2g	LSMSA, unbiased, KTH/LM	0.0305	0.0175	0.0145	0.0257	0.0219	0.0336	0.0361	0.0193	No ice corr, small area
DTU	3a	R-C-R, RTM, W&G 90-100, GRAVSOFT	0.0418	0.0250	0.0164	0.0196	0.0322	0.0387	0.0404	0.0319	IUGG version
NLS-FGI	4a	R-C-R, RTM, W&G 40-50, GRAVSOFT	~~0.1012~~	~~0.0413~~	~~0.0398~~	~~0.0444~~	~~0.0283~~	~~0.0314~~	~~0.0714~~	~~0.0437~~	Error discovered, will be updated soon
NMA	5b	R-C-R, RTM, W&G 140-150, GRAVSOFT	0.0352	0.0226	0.0192	0.0243	0.0210	0.0350	0.0394	0.0268	Version March 2016
	5c	R-C-R, RTM, W&G 165-175, GRAVSOFT	0.0353	0.0181	0.0198	0.0234	0.0314	0.0363	0.0413	0.0243	Version March 2016

Testing results per country

(meter)		Before fit	After a common 1-parameter fit of the final gravimetric model		
	#	Mean	Mean	StdDev	RMS
All	**2538**	**-0.4874**	**0.0000**	**0.0285**	**0.0285**
Denmark	675		**-0.0184**	0.0168	0.0249
Estonia	114		**0.0068**	0.0147	0.0161
Finland	50		**0.0049**	0.0215	0.0218
Latvia	54		**-0.0188**	0.0246	0.0308
Lithuania	546		**0.0020**	0.0333	0.0333
Norway	902		**0.0087**	0.0285	0.0298
Sweden	197		**0.0177**	0.0186	0.0256

Comparison with other geoid models

Model	Standard deviation in 1-parameter fit (meter)							
	All	Denmark	Estonia	Finland	Latvia	Lithuania	Norway	Sweden
NKG2015	0.0285	0.0168	0.0147	0.0215	0.0246	0.0333	0.0285	0.0186
NKG1996	0.0907	0.0305	0.0356	0.0737	0.0240	0.0308	0.1078	0.0499
NKG2004	0.0908	0.0274	0.0362	0.0367	0.0782	0.0418	0.0698	0.0431
EGG08	0.0436	0.0198	0.0238	0.0201	0.0336	0.0389	0.0537	0.0253
EGM2008 to 2190	0.0468	0.0227	0.0361	0.0577	0.0285	0.0299	0.0597	0.0287
EIGEN-6C4 to 2190	0.0421	0.0216	0.0341	0.0436	0.0292	0.0366	0.0503	0.0283

Difference between NKG2015 and GNSS/levelling